

Ernst & Sohn



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ESSENTIALS OF JUNIOR HIGH SCHOOL MATHEMATICS

BOOK THREE

BY

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PREFACE

THE "Essentials of Junior High School Mathematics" consists of three books planned as a unit to cover the work of the seventh, eighth, and ninth school years. The content and organization of the material have been determined largely by the recommendations of the National Committee on Mathematical Requirements and by a careful consideration of recent surveys and courses of study.

During the three years covered by these books the pupil is undergoing important physical and mental changes. His outlook on life is growing broader and his interests and abilities are expanding. The course in Junior High School mathematics should keep pace with these changes. It should be planned with a view to appealing to the new interests of the pupil and to supplying his new needs.

The aims of this Junior High School Course in mathematics may be summarized as follows:

(1) To give the pupil such a mastery of elementary number combinations as will enable him to perform with skill all common numerical operations.

(2) To develop his ability to think clearly and judge soundly and to cultivate his powers of generalization.

(3) To afford a clear insight into the quantitative and spatial relations of the objects by which he is surrounded.

(4) To produce skill in the application of mathematical principles to the problems of everyday life.

(5) To give the pupil an idea of the tools of mathematics and of the breadth of their application.

BOOK THREE places the chief emphasis on *algebra*. The approach to this subject is made easy by correlating it with arithmetic and with the formula and by anticipating and guarding against the pitfalls that confront the beginner. The work throughout is in accord with the principles of modern psychology.

The difficulties of factoring and algebraic fractions are postponed till the second half of the year. The importance of the functional relation is duly emphasized. The frequent drills and reviews help to clinch the principles and to form correct habits.

In spite of its simplicity, this book covers all the requirements for elementary algebra of the College Entrance Examination Board and of the National Committee on Mathematical Requirements.

The book abounds in geometric concepts and applications of geometry to algebra. In addition, a brief introduction to *demonstrative geometry* is included in order to familiarize the pupil with the nature of geometric proofs. An introduction to *trigonometry* further widens the pupil's mathematical horizon.

The *problems* are related to the pupil's interests and deal, so far as practicable, with real situations. Puzzling problems and algebraic fallacies are given at the end of several chapters to interest the inquiring student in somewhat more difficult work. A chapter of exercises that supplement and review the work of the entire year is included at the end of the book.

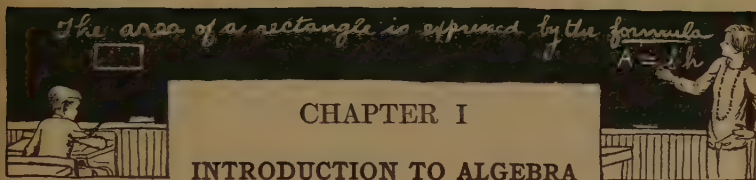
Provision for *individual differences* among pupils is made by supplementing the many simple exercises and problems intended for the average pupil by sufficient material to keep the brightest pupils active.

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CHAPTER I

INTRODUCTION TO ALGEBRA

An Extension of Arithmetic

You have learned how important a part arithmetic plays in all the affairs of life. In your games and in your studies, at home and outside the home, you are constantly brought face to face with problems that call for a knowledge of the principles of arithmetic. What was John's batting average? What per cent did Mary have on her last arithmetic test? What change should the clerk hand you? How much must you save each day for that coveted radio set? Is your gas bill correct? Can your father afford a new car? These and hundreds of other questions call daily for a knowledge of arithmetic.

You are now about to study another branch of mathematics called **algebra**. This study will greatly increase your facility in many kinds of computations and will enable you to grasp clearly and attack successfully some of the more complex problems that press for solution from time to time.

Algebra has been aptly termed the "shorthand" of arithmetic because it greatly shortens the work. It might also be called "generalized arithmetic," since it uses letters to represent all possible values instead of confining itself to numbers of definite values. By this means algebra reveals many truths that lie concealed in arithmetic and might easily be overlooked.

Symbols of algebra. Many of the principles of arithmetic apply also to algebra. The signs $+$, $-$, $\times$, $\div$, and $=$ are used in algebra with the same meaning as in arithmetic, although, as you will see later, the signs $+$ and $-$ have another meaning as well.

Multiplication in algebra may be indicated by a raised dot or by writing several letters side by side. Thus, $5 \times a$ may be written $5 \cdot a$ or $5a$, $5 \times a \times b$ may be written $5 \cdot a \cdot b$ or $5ab$. In general, if the numbers are represented by letters, all signs of multiplication are omitted.

As in arithmetic $3 \times 5 = 5 \times 3$, so in algebra $ab = ba$.

A **factor** of a product is one of the numbers that multiplied together give the product. Thus, 5, a , and b are factors of the product $5ab$.

Using Formulas

Suppose that you wish to find the cost of 3 baseball bats at \$2 each. You know that you must multiply the price of 1 bat by the number of bats. The statement of this rule may be abbreviated to:

The cost = the number $\times$ the price

By using the letter c to represent *any number* of units in the cost, n to represent *any number* of bats, and p to represent *any number* of units in the price, the rule may be shortened to read:

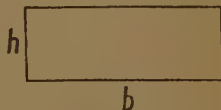
$$c = np$$

In the case of 2-dollar bats, you have $c = 2n$.

What would c equal if each of 3 bats cost \$1? \$1.25? \$1.50?

The area of a rectangle is found by multiplying the number of units in the base by the number of units in the height, or altitude. This is commonly expressed thus:

$$\begin{aligned} \text{Area} &= \text{base} \times \text{height, or} \\ A &= bh \end{aligned}$$



Such abbreviated statements are called "formulas." A **formula** is a rule for working a problem, expressed in algebraic symbols.

In the formula $A = bh$, b and h are called **general numbers**, since they represent any numbers that may be substituted for the letters of the formula.

EXERCISES

1. When the price of radio tubes is \$3 each, what is the cost of 2 tubes? of 5 tubes? of n tubes?

The cost of n tubes gives the formula $c = 3n$.

2. Using the formula derived in ex. 1, (a) find the cost when $n = 1$; (b) find the cost when $n = 5$.

3. Find the cost of 4 tubes at \$2.50 each; at \$5 each; at d dollars each. What is the formula for the cost of 4 tubes at d dollars each?

4. Using the formula derived in ex. 3 ($c = 4d$), find the cost of 4 tubes when $d = \$3$; when $d = \$4.50$.

5. How much will n tubes cost at d dollars each? Find the cost of tubes when $n = 2$ and $d = \$2$; when $n = 8$ and $d = \$2.50$.

6. If $M = 1000$ and $C = 100$, what is the value of $3M + 8C$? of $4M + 3C$? of $2M - C$?

7. If $q = 25$ cents and $d = 10$ cents, what is the value of $q + d$? of $2q + 3d$? of $5q + 2d$?

8. What is the area of a rectangle having a height of 6 units and a base of 5 units? the same height with a base of 3 units? of b units?

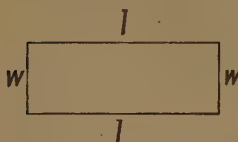
9. State the formula for the area of a rectangle in terms of the base and the height. Find the area when $h = 6$ and $b = 4$.

10. Find the area of a rectangle with a base of 7 units and a height of 2 units; with the same base and a height of 6 units; of h units.

11. Find the area of a rectangle when $b = 7$ and $h = 3$; when $b = 12$ and $h = 15.5$.

12. If $a = 5$ and $b = 6$, find the value of $5a + 8b$; of $7a - b$.

13. If $a = 3$ and $b = 7$, what is the value of $2a + 2b$ of $3ab$?



14. The **perimeter** of a figure is the distance around it. Write a formula for the perimeter, p , of the parallelogram shown at the left. Find p when $l = 8$ in.

and $w = 3$ in.; when $l = 2.5$ ft. and $w = 1.5$ ft.

15. What is the perimeter of a rectangular city block 405 ft. long and 200 ft. wide? Draw a figure to represent this.

16. A farmer has a rectangular meadow that is 25 rd. wide and 32 rd. long. What is its perimeter?

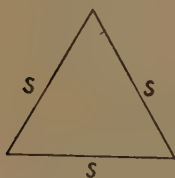
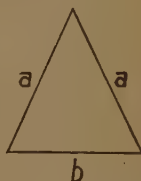
17. Draw a diagram to represent a triangle. Letter the three sides a , b , and c . Write a formula for its perimeter.

18. Find the perimeter of a triangle if $a = 8$ in., $b = 7$ in., and $c = 12$ in.

19. Find the perimeter of a triangle if $a = 3$ ft., $b = 4$ ft., and $c = 5$ ft.

20. Howard ran around a triangular course whose sides were 408 ft., 572 ft., and 620 ft., respectively, at the rate of 16 ft. a second. How long did it take him?

21. A triangle having two sides equal is called an **isosceles triangle**. How can you express its perimeter in terms of a , one of the equal sides, and b , the base?



22. What is the perimeter of an isosceles triangle when $a = 10$ in., $b = 15$ in.? when $a = 2\frac{1}{2}$ ft., $b = 3$ ft.?

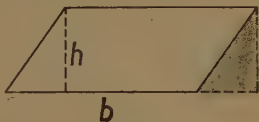
23. A triangle having three sides equal is called an **equilateral triangle**. Express its perimeter in terms of its side, s .

24. Find the perimeter of an equilateral triangle when one side is 2 in. ; 8 in. ; $12\frac{1}{2}$ in.

The **height**, or **altitude**, of a parallelogram is the line drawn between the upper and the lower base perpendicular to the bases.

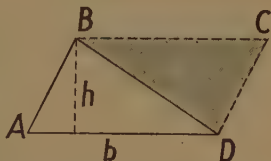
The area of a parallelogram is equal to the product of the number of units in the base and the height.

That is, $A = bh$



25. Find the area of a parallelogram when $b = 5$ in. and $h = 3$ in. ; when $b = 7$ in. and $h = 5$ in. ; when $b = 4$ ft. and $h = 2$ ft.

The diagonal BD of the parallelogram at the right divides it into two equal triangles. This may be shown by cutting out a parallelogram and then cutting along one diagonal. The two triangles formed can be made to fit exactly over each other.



The **height**, or **altitude**, of a triangle is the line drawn from the vertex of the triangle perpendicular to the base.

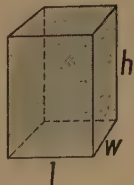
Since each of these triangles is one half of the parallelogram, the area of each triangle is one half of the area of the parallelogram.

The area of a triangle equals one half the product of the number of units in the base and the altitude.

That is, $A = \frac{1}{2}bh$

26. Find the area of a triangle when $b = 8$ in. and $h = 4$ in. ; when $b = 3$ ft. and $h = 2\frac{1}{2}$ ft.

27. The volume of a rectangular solid is represented by the formula $V = lwh$. Find the volume of a rectangular solid when $l = 3$ in., $w = 2$ in., $h = 5$ in. ; when $l = 8$ in., $w = 5$ in., $h = 6$ in. ; when $l = 15$ in., $w = 3$ in., $h = 4$ in.



Expressing Division

If you want to write 16 divided by 8, you may write it in two ways, thus: $16 \div 8$ or $\frac{16}{8}$. Similarly, you may write a divided by b as $a \div b$ or $\frac{a}{b}$. The form $\frac{a}{b}$ is the fractional form and, as you will remember from arithmetic, the upper number is called the **numerator** and the lower number, the **denominator**. One half of a may be written either $\frac{1}{2}a$ or $\frac{a}{2}$.

EXERCISES

Find the value of $\frac{a}{b}$ when:

1. $a = 12, b = 4$

3. $a = 18, b = 6$

2. $a = 12, b = 3$

4. $a = 18, b = 12$

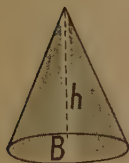
If velocity is indicated by V , distance by d , and the time by t , what is the meaning of the formula $V = \frac{d}{t}$? Find V when:

5. $d = 20$ ft., $t = 2$ sec.

7. $d = 2000$ ft., $t = 4$ min.

6. $d = 20$ mi., $t = 3$ hr.

8. $d = 240$ mi., $t = 8$ hr.



9. The volume of a cone is found by substituting for B and h in the formula $V = \frac{1}{3}Bh$. Find V when $h = 12$, $B = 150$; when $h = 8$, $B = 72$.

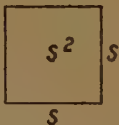
10. The formula for the sum, s , of two fractions, $\frac{1}{a}$ and $\frac{1}{b}$, is $s = \frac{b+a}{ab}$. Test this to see if it is true for $\frac{1}{2} + \frac{1}{3}$.

11. Find s if $a = 4$ and $b = 5$; if $a = 3$ and $b = 4$.

12. One of the formulas used in electricity is $C = \frac{E}{R}$. Find C when $E = 550$, $R = 10$; when $E = 110$, $R = 22$.

Powers, Exponents, Coefficients

You remember that when the four sides of a rectangle are all equal, the figure is called a **square**. If s represents the length of one of the sides, the area of the square is $s \times s$, or s^2 . This is read s square.



You know also that when the length, the width, and the height of a rectangular solid are all equal, the solid is called a **cube**. The volume of a cube whose side is s units long is $s \times s \times s$, or s^3 . This is read s cube.

A product, like s^3 , that is made up of a number of equal factors is called a **power** of that factor. The factor is called the **base**.



The small figure ^{3} is called an **exponent**. It shows the number of times the base is to be taken as a factor. An exponent is written to the right of and a little above the base.

Any number expressed in algebraic symbols is called an **algebraic expression**. Thus, $5ax$ and $3ax^2 - 2y^3$ are algebraic expressions.

If an algebraic expression can be separated into two factors, either factor is called the **coefficient** of the other factor. Thus, in the expression $5ax$, $5a$ is the coefficient of x , while 5 is the **numerical coefficient** of ax . If nothing is said to the contrary, the term *coefficient* indicates the numerical coefficient.

The coefficient 1 is generally omitted. Thus, a is written for $1a$ and bxy for $1bxy$.

ORAL EXERCISES

1. What is the area of a square if one side is 3 in.? 8 ft.?
2. What sort of figure is one of the faces of a cube?
3. Find the area of a face of a cube if the edge is 7 in.
4. Find the total surface of a cube if the edge is 4 in.

Using Parentheses

If you want to find the sum of 3 times 5 and 3 times 8, you may obtain it at once by finding 3 times the sum of 5 and 8.

$$\begin{aligned} 3 \times 5 + 3 \times 8 &= 3(5 + 8) \\ 15 + 24 &= 3(5 + 8) \\ 15 + 24 &= 3 \times 13 \\ 39 &= 39 \end{aligned}$$

Similarly, $3(a + b) = 3a + 3b$.

The signs () are called **parentheses**. They indicate that the expression included within them is to be considered as one number. Therefore, if possible, *operations within a parenthesis should always be performed first.*

EXERCISES

Read each of the following expressions, tell what each one means, and write it without the use of exponents:

1. $8a^2x^3$ SOLUTION. $8a^2x^3 = 8aaxxx$

2. $7a^2x^2y$

4. $6x^2y^2z$

6. $3mn^2x^5$

3. $12ax^3y^2$

5. $15a^4b^3x$

7. $5a^2n^3$

Express each of the following in an abbreviated form:

8. $x + x + x + x$

11. $a^3 + a^3 + b^2 + b^2 + b^2$

9. $xxxx$

12. $3abbb$

10. $2 \times 2 \times 2 aabb$

13. $5aaxxy$

14. Why is the second power of a number called the *square* of the number? Give a reason for calling the third power of a number the *cube* of the number.

State the coefficient of x in each of the following:

15. $6ax$

16. $5abx$

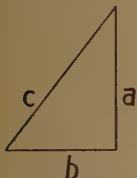
17. $\frac{1}{2}ab^2x$

18. $\frac{2}{3}(a+b)x$

19. State the exponents and the numerical coefficients in ex. 1 to 7.

When $a = 2$, $b = 3$, $n = 4$, $x = 2$, and $y = 0$, find the value of :

- | | | |
|--------------|-------------------|----------------------|
| 20. a^2 | 27. $3(an)^2$ | 34. $ax(a + y)^2$ |
| 21. a^3 | 28. $(3an)^2$ | 35. $5a^2 + (5a)^2$ |
| 22. $3a^3$ | 29. $5ab^3$ | 36. $(nx)^2 - nx^2$ |
| 23. $2a^2$ | 30. $a(b + n)$ | 37. $(x + y)^2 - xy$ |
| 24. $(2a)^2$ | 31. $ab(n + y)$ | 38. $abxy$ |
| 25. $5an$ | 32. $xy(a^2 + b)$ | 39. $a^2(x^2 - y^2)$ |
| 26. $3an^2$ | 33. $a^3(2a - n)$ | 40. $(3b)^2 - 3b^2$ |

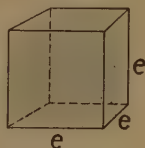


The square of the longest side of a right triangle (the **hypotenuse**) is equal to the sum of the squares of each of the other two sides. That is, $c^2 = a^2 + b^2$.

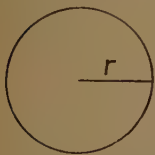
41. Find c in a right triangle when $a = 3$, $b = 4$; when $a = 8$, $b = 6$; when $a = 12$, $b = 5$.

42. State the formula for the volume of a cube. Find the volume when the edge is 3 in.; when it is 2.3 ft.

43. The six faces of a cube are equal squares. If one edge of the cube is e , write a formula for the total surface. Find the total surface when $e = 1$ in.; when $e = 2$ in.; when $e = 5$ ft.



44. The circumference of a circle is given by the formula $C = 2\pi r$. r is the radius and π is the approximate number $3\frac{1}{7}$. Find the circumference of a circle when $r = 1$; when $r = 3.5$.



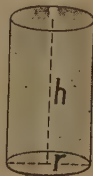
NOTE. π is often expressed by the approximate decimal

3.1416; but in this book $3\frac{1}{7}$ is to be used unless another value is specified.

45. Find the circumference of an automobile wheel if its radius is 15 in.

46. The area of a circle is given by the formula $A = \pi r^2$. Find A when $r = 4.2$; when $r = 5\frac{3}{5}$.

47. The volume of a cylinder is found by using the formula $V = \pi r^2 h$. Find V when $r = 3.5$ in. and $h = 4$ in.; when $r = 5$ in. and $h = 4.2$ in.





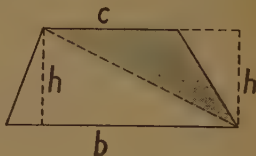
48. The formula $S = 4 \pi r^2$ gives the surface of a sphere in terms of its radius. Find S when $r = 3\frac{1}{2}$ in.; when $r = 14$ in.

49. What is the approximate surface of the earth if r is taken as 4000 mi.?

A diagonal of a trapezoid divides the trapezoid into two triangles. The area of the trapezoid is equal to the sum of the areas of the two triangles. That is,

$$A = \frac{1}{2}hb + \frac{1}{2}hc, \text{ or } A = \frac{1}{2}h(b + c)$$

50. Find the area of a trapezoid if $h = 2$ in., $b = 5$ in., and $c = 3$ in.; if $h = 7$ in., $b = 12$ in., and $c = 6$ in.



51. The formula $S = \frac{1}{2}at^2$ is used in physics to determine the distance a freely falling body will drop in t seconds. If $a = 32$, find S when $t = 1$; when $t = 2$; when $t = 3$.

52. The horsepower of some gasoline engines is given by the formula $P = \frac{nd^2}{2.5}$, in which d is the diameter of the cylinder in inches and n is the number of cylinders. What is the rated horsepower of a 6-cylinder engine when $d = 3.25$ in.?

Summary of important formulas.

- | | |
|--|-----------------------------|
| 1. The area of a parallelogram or of a rectangle | $A = bh$ |
| 2. The area of a triangle | $A = \frac{1}{2}bh$ |
| 3. The area of a square | $A = s^2$ |
| 4. The area of a circle | $A = \pi r^2$ |
| 5. Velocity formula | $V = \frac{d}{t}$ |
| 6. The perimeter of a parallelogram | $p = 2l + 2w$ |
| 7. The circumference of a circle | $C = \pi d$ or $C = 2\pi r$ |
| 8. The volume of a cube | $V = s^3$ |
| 9. The volume of a rectangular solid | $V = lwh$ |
| 10. The hypotenuse of a right triangle | $c^2 = a^2 + b^2$ |

The Language of Algebra

Before we can solve problems by algebra, we must translate them into statements with algebraic symbols.

General problems can be represented as formulas. Others can be put into the shorthand of algebraic language by following the statement of the problem.

For example, if Mary had c cents and Jane had 10 cents more than Mary, Jane had $(c + 10)$ cents and the number of cents both together had was $c + c + 10$, or $2c + 10$.

EXERCISES

1. George earned \$3.50 a day and his father earned twice as much. How much did his father earn? How much did both earn?

2. Frank earned d dollars a day and his father earned twice as much. How much did his father earn? How much did the two earn in a week of 6 days?

3. Tom had c cents. How much did James have if he had 18 cents more than Tom? How much did both have?

4. Harriet had k cents and Nina had c cents more than Harriet. How much did Nina have? How much did both have?

5. There were n girls in the biology class and 5 fewer boys than girls. How many boys were there in the class? What was the total number of pupils in the class?

6. The altitude of a triangle is h inches. What is the base of the triangle if it is 3 in. more than $\frac{1}{2}$ of the altitude?

7. If John had d dollars and saved n dollars a week for 2 weeks, how much did he then have?

8. Julia had b books. If she received n books for Christmas and 3 books for her birthday, how many did she then have?

9. What is the perimeter of a square whose side is s ?

10. If 3 baseballs cost d dollars, how much will 1 ball cost? 5 balls? n balls?

11. When eggs cost 48¢ a dozen, how much will 1 egg cost? 2 dozen eggs? 9 eggs? n eggs? d dozen eggs?

12. Suppose that eggs are c cents a dozen, how much will 1 egg cost? n eggs? d dozen eggs?

13. Mr. Clark drives his automobile at the rate of m miles an hour. At that rate, how far will he go in 2 hr.? in 3 hr.? in 15 min.? in h hours?

14. How long will it take Mr. Clark to drive 10 mi. at the rate of m miles an hour? n miles an hour? $5n$ miles an hour?

15. Harry had d dollars. How many cents did he have? He spent c cents for lunch, n cents for car fare, and 5 cents for a pencil. How many cents did he have left? Express this result in dollars.

16. Mr. Kane is 45 yr. old. How old was he 5 yr. ago? y years ago? If Mr. Kane is k years old, express his age y years ago; n years hence.

Consecutive numbers are numbers that follow each other in the natural order of counting, as 5, 6, 7.

17. Write five consecutive numbers having 20 for the first number; having 12 for the middle number; having n for the middle number; having 15 for the largest number.

18. What is an even number? an odd number? State which of the following numbers are even and which are odd:

8 17 19 21 22 23 24 26 28 30 37

19. Consecutive odd numbers and consecutive even numbers follow each other in the order of counting, as 1, 3, 5 and 2, 4, 6. In ex. 18, find three consecutive odd numbers; four consecutive numbers; four consecutive even numbers.

20. If n is an integer (whole number), write an even number; an odd number.

Solving Easy Equations

If you are told that 3 notebooks can be bought for 21 cents, you may think of this problem in this form: "The price of 3 notebooks = 21 cents," or, in algebraic language, in this form:

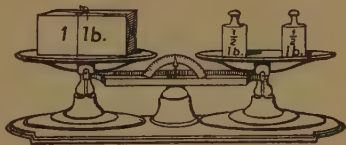
$$3b = 21$$

Such a statement is called an "equation."

An **equation** is a statement that two quantities are equal to each other. The quantity on the left side of the equality sign is called the **first member**, or the **left member**, and the other quantity is called the **second member**, or the **right member**, of the equation.

Some expressions that look like equations are not true equations. For example, the statement $n + 5 = n$ is not an equation, since it is impossible for a number increased by 5 to equal itself.

An equation may be compared to balanced scales, the two pans of which represent the two members of the equation. To maintain the balanced condition of the scales, any change that is made on one side of the scales must be balanced by an equivalent change on the other side.



BALANCED SCALES

Suppose that the scales are balanced in weighing a box of candy. To keep them balanced, if a clerk takes out half a pound of candy from the box in one pan, he must remove a half-pound weight from the other pan. If he adds a pound of candy to the box, he must add a pound weight to the other pan. If he puts only one half the amount of candy in one pan, he should have only half the weights in the other pan, and if he puts double the amount of candy in one pan, he should double the weights in the other pan.

Golden rule of the equation. *Whatever you do to one side of an equation, do also to the other side.*

If you wish to find the value of b in the equation $3b = 21$, proceed as follows:

Statement of the equation:

$$3b = 21$$

As b is $\frac{1}{3}$ of $3b$, divide both sides by 3:

$$\frac{3b}{3} = \frac{21}{3}$$

$$\text{or } b = 7$$

An equation is **solved** when a value of the unknown quantity has been found which, if substituted in the equation, will make both sides of the equation the same.

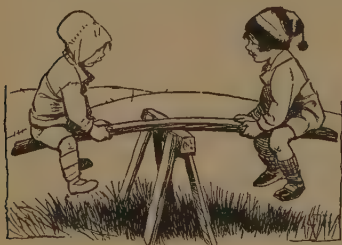
This value is called a **root** of the equation and is said to *satisfy* or fulfill the conditions of the problem. In the case above, $b = 7$ is a root of the equation. It satisfies the condition that $3b = 21$, since $3 \times 7 = 21$.

All values found for the unknown quantities should be substituted in the *original* equation to see if they satisfy the equation. This is called a **check** of the work.

EXERCISES

1. Which of the values $x = 5$, $x = 8$, or $x = 12$ satisfies the equation $2x - 3 = x + 9$?

2. Is the equation $\frac{2}{3}x + 9 = \frac{1}{4}x + 19$ satisfied by $x = 12$? by $x = 4$? by $x = 24$?



3. What values of x will satisfy the equation $3x = 18$? How many such values do you find?

4. How may a seesaw, balanced as in the figure, be used to illustrate an equation?

Read the following equations and translate them into word problems. Then solve the equations and check the results.

5. $3n = 18$

HINT. If 3 times a certain number is 18, what is the number?

- | | | |
|----------------------|-----------------------|----------------------|
| 6. $2x = 24$ | 10. $12a = 180$ | 14. $5x - x = 24$ |
| 7. $6x = 42$ | 11. $9n = 27$ | 15. $2a + 3a = 30$ |
| 8. $5n = 20$ | 12. $5n - 2n = 21$ | 16. $13b - 5b = 56$ |
| 9. $15x = 240$ | 13. $8n + 3n = 55$ | 17. $25x + 3x = 112$ |
| 18. $2x + 7x = 0$ | 21. $2.3x + .7x = 30$ | |
| 19. $7y - 2y = 22.5$ | 22. $28n - 15n = 39$ | |
| 20. $3x + 15x = 54$ | 23. $n + 5n = 42$ | |

24. Douglas and Lee have 42 marbles between them. Lee has twice as many marbles as Douglas. How many has each?

SOLUTION.

Let m = the number of marbles Douglas has
 Then $2m$ = the number of marbles Lee has
 and $3m$ = the number of marbles both have
 But 42 = the number both have

Therefore $3m = 42$
 Divide both sides by 3: $m = 14$
 $2m = 28$

CHECK. Check in the original problem:

$$2 \times 14 = 28, 14 + 28 = 42$$

NOTE. The horizontal bar separates the equation of the problem from the algebraic statements on which it is based.

25. Mary and Anna together have 84¢. How much has each if Mary has 3 times as much as Anna?

26. How much money has each if Mary has twice as much as Anna and together they have 84¢?

27. A man and his son together earned \$13.50. How much did each receive if the father earned twice as much as the son?

28. There are 36 students in a classroom in algebra. If there are 5 times as many boys as girls, how many are there of each?

29. The perimeter of a triangle is 24. Find each side if one side is 3 times as long as the shortest side and the other side is equal to the sum of the first two sides.

HINT. Let one side = s . Then the other two sides will be $3s$ and $4s$; hence $s + 3s + 4s = 24$.

Axioms. In solving equations, you have made use of the fact that both sides of the equation may be divided by the same number without changing the equality. A statement of a principle accepted as true without proof is called an **axiom**.

Division axiom. *Both members of an equation may be divided by the same number without destroying the equality.*

NOTE. Division by zero must be excepted.

EXAMPLE 1. Solve the equation $5x + 3 = 18$.

SOLUTION. It is evident that 3 more than $5x$ is 18. Hence you may find the value of $5x$ by subtracting 3 from both sides of the equation, thus:

$$5x + 3 - 3 = 18 - 3$$

$$\text{or } 5x = 15$$

Divide both sides by 5:

$$\frac{5x}{5} = \frac{15}{5}$$

$$\text{or } x = 3$$

CHECK. Substitute 3 for x :

$$5(3) + 3 = 18$$

$$15 + 3 = 18$$

$$18 = 18$$

EXAMPLE 2. Solve $3x + 5 = x + 17$.

SOLUTION.

$$3x + 5 = x + 17 \quad (1)$$

* Subtract x from both sides of (1): $3x + 5 - x = x + 17 - x$ (2)

$$\text{or } 2x + 5 = 17 \quad (3)$$

* Subtract 5 from both sides of (3): $2x = 12$ (4)

Divide both sides of (4) by 2:

$$x = 6$$

NOTE. The steps marked * could be combined into one step by subtracting $x + 5$ from each side of the equation. This would shorten the work, thus:

$$3x + 5 - x - 5 = x + 17 - x - 5 \quad (2)$$

$$2x = 12 \quad (3)$$

$$x = 6$$

CHECK.

$$3(6) + 5 = 6 + 17$$

$$18 + 5 = 23$$

$$23 = 23$$

Subtraction axiom. *The same number may be subtracted from both sides of an equation without destroying the equality.*

EXERCISES

Solve and check each of the following exercises :

- | | |
|----------------------|----------------------------|
| 1. $n + 8 = 18$ | 8. $4x + 15 = 75$ |
| 2. $3 + y = 17$ | 9. $8s + 18 = 3s + 38 + s$ |
| 3. $7 + x = 15$ | 10. $8 + 2x = 18x$ |
| 4. $2n = n + 3$ | 11. $24 + x + 6 = 7x$ |
| 5. $2n + 5 = n + 15$ | 12. $8y + 15 = 6y + 33$ |
| 6. $3n = 52 + n$ | 13. $5n + 3 = 2n + 19 + n$ |
| 7. $5x = 15 + 2x$ | 14. $5x + 15 - x = 48 + x$ |

15. Read ex. 1 to 7 and state each one orally as a word problem. Thus, ex. 1 may be stated, "What number increased by 8 is equal to 18?"

16. Mr. Drake divided \$5 between his two sons so that the older boy had \$1 more than the younger. How much did each receive?

HINT. Let d = the number of dollars the younger boy received
 $d + 1$ = the number of dollars the older boy received

Then $2d + 1 = 5$, the number of dollars the two boys received

17. John and Carl received \$23 for a week's wages. If John received \$5 more than Carl, how much did each earn?

HINT. $c + (c + 5) = 23$. Explain.

18. Mary spent 5¢ more than twice as much as Jane did for lunch. If they spent 50¢ together, how much did each spend?

19. Find two consecutive numbers whose sum is 37.

HINT. If n is the smaller number, $n + 1$ is the larger number.

20. Find two consecutive even numbers whose sum is 46.

HINT. $n + (n + 2) = 46$, or $2n + 2 = 46$. Explain.

21. Find three consecutive odd numbers whose sum is 51.

22. One number is 5 more than a second number and their sum is 41. Find each number.

23. Read ex. 16 to 18 and state each as a number problem. Thus, ex. 16 may be stated, "Divide 5 into two parts such that one part is 1 greater than the other part."

EXAMPLE 1. Solve the equation $m - 8 = 15$.

SOLUTION. The equation states that 8 less than the number m is 15. Hence to find m , add 8 to both sides of the equation, thus:

$$\begin{aligned} m - 8 + 8 &= 15 + 8 \\ \text{or } m &= 23 \end{aligned}$$

CHECK.

$$\begin{aligned} 23 - 8 &= 15 \\ \text{or } 15 &= 15 \end{aligned}$$

Addition axiom. *The same number may be added to both sides of an equation without destroying the equality.*

EXAMPLE 2. Solve the equation $\frac{3}{4}x = 9$.

SOLUTION.

$$\frac{3}{4}x = 9 \quad (1)$$

Multiply both sides of (1) by 4: $4 \times \frac{3}{4}x = 4 \times 9 \quad (2)$

$$\text{or } 3x = 36 \quad (3)$$

Divide both sides of (3) by 3: $x = 12 \quad (4)$

CHECK.

$$\begin{aligned} \frac{3}{4}(12) &= 9 \\ 9 &= 9 \end{aligned}$$

What was the reason for the first step of the solution, namely, "Multiply both sides of (1) by 4"?

Multiplication axiom. *Both sides of an equation may be multiplied by the same number without destroying the equality.*

Summary of Principles.

The four axioms used in solving equations are four principles that are united into one general law of the equation that we have called the *Golden rule of the equation*. (See page 14.)

1. If a number is added to one side of an equation, an equal number must be added to the other side of the equation.

2. If a number is subtracted from one side of an equation, an equal number must be subtracted from the other side of the equation.

3. If one side of an equation is multiplied by a number, the other side of the equation must be multiplied by the same number.

4. If one side of an equation is divided by a number, the other side of the equation must be divided by the same number.

EXERCISES

Solve and check the following equations:

1. $n - 12 = 28$

7. $\frac{1}{2}x - 2 = 8$

2. $18 = n - 2$

8. $\frac{1}{3}n - 6 = 5$

3. $3x - 5 = 15 - x$

9. $5n - 6 = 8 - 2n$

4. $2x - 15 = 9 - x$

10. $n - 5 = \frac{3}{4}n + 2$

5. $2x - 7 = 19 + x$

11. $2n + 3 = \frac{1}{3}n + 18$

6. $x = \frac{1}{2}x + 8$

12. $3n - 2 = 2\frac{1}{2}n + 1$

13. $\frac{n}{2} = 8$

14. $\frac{y}{3} = 6$

15. $36 = \frac{x}{2}$

16. $18 = \frac{x}{4}$

17. Read ex. 5 to 10 and state each as a word problem.

18. Two boys caught 18 fish. One boy caught 4 fish less than the other. How many fish did each catch?

HINT. Let n = the number of fish the first boy caught

Then $n - 4$ = the number the second boy caught

Hence $2n - 4 = 18$

19. Fred and Chester spent 30¢ for notebooks. How much did each spend if Fred spent 6¢ less than Chester?

20. In Miss Baird's history class there were 6 more girls than boys. If there were 32 pupils in the class, how many girls were there? how many boys?

21. Jack is 25 yr. younger than his father and the sum of their ages is 65 yr. How old is each?

22. Mary is 5 yr. younger than her brother Paul. How old is each if the sum of their ages is 21 yr.?

23. Louis and George paid \$85 for two radio sets. Louis paid \$20 less than twice as much as George paid. How much did each pay?

24. Read ex. 18 to 23 as number problems. Thus, ex. 18 may be read, "The sum of two numbers is 18. What are the numbers if one number is 4 less than the other number?"

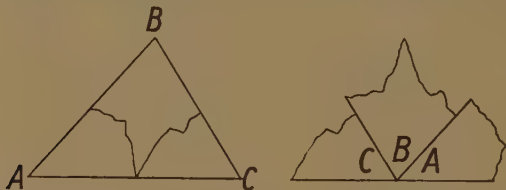
25. How much do eggs cost per dozen when $\frac{1}{3}$ of a dozen cost 30¢ less than a dozen cost?

HINT. Let c = the number of cents eggs cost per dozen
Then $\frac{1}{3}c = c - 30$

26. How much do oranges cost per dozen when $\frac{2}{3}$ of a dozen cost 40¢?

27. Arthur bought a suit of clothes and a pair of shoes for \$49. If the shoes cost $\frac{1}{6}$ as much as the clothes, what was the price of each?

28. Draw a triangle similar to triangle ABC below. Cut off the angles A and C and place them next to angle B , as shown in



the diagram. You will find that the sum of the three angles of the triangle is equal to one straight angle, or 180° .

29. Find each angle of a triangle if angle C is equal to angle B and angle A is equal to the sum of angle B and angle C .

30. Find each angle of a triangle if angle B is 4 times as large as angle A and angle C is equal to the difference between angle B and angle A .

31. Two angles of an isosceles triangle are equal to each other. Find the size of each angle of an isosceles triangle if the equal angles are each 4 times as large as the third angle.

REVIEW EXERCISES

1. What are general numbers? What advantage is gained by their use? Explain.

2. We may indicate $a \times b$ by ab . Explain why we cannot indicate 7×3 in the same way.

3. State the exponents and the coefficients in $7ax^2y^3 - 5b^4x^n$.

If $a = 3$, $b = 2$, $x = 2$, $y = 1$, $z = 0$, find the value of:

- | | | |
|-------------|------------|-------------------|
| 4. $2x^2$ | 7. $3a^2y$ | 10. $2a^2 + yz$ |
| 5. $(3x)^2$ | 8. $5y^4$ | 11. $b(a^2 + 2z)$ |
| 6. $2a^4$ | 9. axz^2 | 12. $3z(4b + a)$ |

13. Express each of the following formulas in words telling what it represents:

- | | |
|-------------------------|-------------------------|
| (1) $c = np$ | (6) $c^2 = a^2 + b^2$ |
| (2) $A = bh$ | (7) $C = \pi d$ |
| (3) $d = Vt$ | (8) $A = \pi r^2$ |
| (4) $A = \frac{1}{2}bh$ | (9) $V = \frac{1}{3}Bh$ |
| (5) $p = 2(l + w)$ | (10) $V = \pi r^2h$ |

14. The cost of sending a telegram of more than 10 words is 45¢ plus 3¢ for each additional word. State this as a formula.

15. Using the formula of ex. 14, find the cost of sending a telegram of 12 words; of 15 words; of 75 words; of 100 words.

16. Using the formula of ex. 14, find the number of words in a telegram if the cost is 48¢; 69¢; 99¢.

17. If i = interest, p = principal, r = rate, t = time in years, and A = amount, state the meaning of $i = prt$; of $A = p + prt$; of $A = p(1 + rt)$.

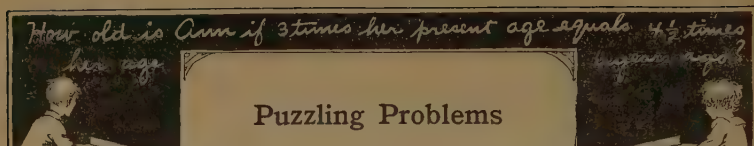
18. Find the interest on a \$500 Liberty bond at $4\frac{1}{4}\%$ for 2 yr.; for 3 yr. 6 mo.; for 5 yr.

19. In what time will the interest on \$2000 at 5% be \$22.50?

20. What is the amount of \$250 for 1 yr. 8 mo. at $4\frac{1}{2}\%$?

21. The base of a rectangle is 3 times its altitude and their sum is 16 in. Find the base, the altitude, and the area.

22. If the altitude of a rectangle is 5 in. longer than the base and their sum is 25 in., find the altitude, the base, and the area.



Puzzling Problems

You will discover how helpful equations are in the solution of the following problems.

1. A bottle and a cork cost \$.50. How much did the bottle cost if it cost \$.40 more than the cork?

2. If a fish and a half a fish weigh a pound and a half, how much will a dozen and a half fish weigh?

3. What is the weight of a dog that weighs 24 lb. plus $\frac{1}{3}$ of its weight?

4. If 3 boys catch 3 fish in 3 min., at the same rate how many boys will be required to catch 24 fish in 12 min.?

5. How old is Ann if 3 times her present age is equal to $4\frac{1}{2}$ times her age 6 yr. ago?

6. Chester is 18 yr. old and Frank is 3 yr. old. In how many years will Chester be 3 times as old as Frank?

7. Fred was asked how many marbles he had. He replied that 22 was 10 more than $\frac{4}{5}$ of the number he had. How many marbles did Fred have?

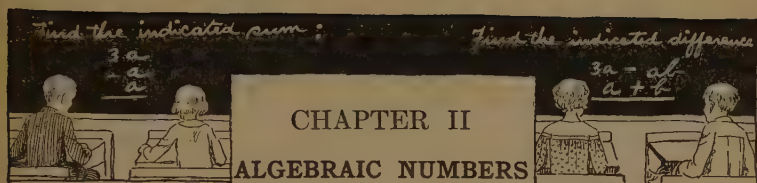
8. The head of a fish is 4 in. long, the tail is as long as the head and $\frac{1}{3}$ of the body, and the body is as long as the head and the tail together. How long is the fish?

9. If a dog and a half could eat a chop and a half in a minute and a half, how many chops could six dogs eat in 6 minutes?

10. The temperature at noon was 32° . By 6 P.M. it had fallen $\frac{1}{3}$ as many degrees as the resulting temperature. What was the temperature at 6 P.M.?

11. A gallon mixture of alcohol and water contains $\frac{1}{4}$ water. How much water must be added to the mixture so that it shall be $\frac{1}{5}$ water?

12. How much pure alcohol should be added to the original mixture in ex. 11 to make the mixture 80% alcohol?



In the last chapter you saw how useful letters are in representing numbers. Rules for working many problems may be stated in the form of short equations or formulas. For example, the rule for finding the cost of a number of articles of the same kind by multiplying the price of one article by the number of articles may be stated briefly as $c = np$. This abbreviated form of expression is one of the great advantages of algebra over arithmetic.

Most of your everyday calculations are with numbers such as you studied in the last chapter and in arithmetic. There are some instances, however, in which numbers must be distinguished in a special way. In playing games, if you win 10 points, you say that you are 10 points "to the good." This you may represent as $+ 10$. If, however, you lose 10 points, you say that you go "in the hole." This you may represent as $- 10$. Again, a boy is credited with the amount he earns and is debited or charged with the amount he owes.

If you are told to find a certain number that when increased by 15 equals 10, you write the equation,

$$n + 15 = 10$$

When you try to solve this equation by subtracting 15 from each side, you meet a difficulty that troubled people for a great many years.

Many ideas about numbers that seem very simple to us caused great difficulty to the early mathematicians. Some of our simplest operations with fractions puzzled the most learned men in the time of Plato and Euclid. There are many mathematical ideas that even to this day it is difficult to grasp. You can easily look out of a window twice, but suppose some one asks you whether you can look out half of a time, what will be your reply? Can you seat 100 pupils in 3 classrooms so that each room will have the same number of pupils?

For many years men thought it impossible to find a number which when added to 15 gives a sum of 10; but we shall soon see how easy it is to find such a number.

Positive and Negative Numbers

To solve problems similar to the one just stated, people found it convenient to divide numbers into two *opposite groups* called "positive numbers" and "negative numbers."

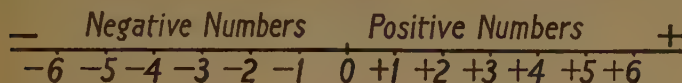
Positive numbers are all the arithmetic numbers and are often distinguished by the sign $+$ written before them, as $+5$, $+a$.

Negative numbers are all the numbers that are opposite in quality or direction to the positive numbers. They are distinguished by the sign $-$ written before them, as -5 , $-a$.

The signs $+$ and $-$ have two meanings. First, they are used as in arithmetic to show the operations of addition and subtraction. Second, they are used to indicate positive and negative numbers. Such numbers are called **algebraic numbers**. When no confusion arises, positive numbers are written like arithmetic numbers with the sign $+$ omitted. The sign $-$, however, is never omitted before a negative number.

The arithmetic value of a number without regard to its direction sign is called the **absolute value**. Thus, $+6$, when regarded as a positive number, is larger than the negative number -6 ; but the absolute value of each is 6.

Positive and negative numbers may be represented graphically, as shown below. If 0 is the starting point, positive numbers are represented by the lengths of the divisions of the line

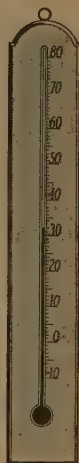


at the right of 0, as $+1$, $+2$, or $+4$. Negative numbers are represented by the lengths of the divisions of the lines at the left of 0, as -2 and -6 .

You are all familiar with the scale of the thermometer. On this scale a temperature has been fixed as the zero or starting point. Degrees above zero may be indicated by positive numbers and those below zero, by negative numbers, as, $+70^\circ$, -10° . The freezing temperature is $+32^\circ$, or 32° above zero, on the Fahrenheit scale.

EXERCISES

1. Draw a straight line scale to represent numbers from -15 to $+15$. On this scale locate the numbers -8 , -3 , 9 , 12 , $8\frac{1}{2}$.
2. Draw a scale to represent thermometer readings from -20° to 100° at intervals of 5° . Can you tell how the zero point is determined on the Fahrenheit thermometer?
3. Locate on the thermometer scale the readings -12° , -4° , 10° , 22° , 70° .
4. If dates after Christ are represented by positive numbers, by what are negative dates represented? What dates would the following be: -350 ? -420 ? -30 ? $+1$? $+1200$? $+1492$? $+1620$?
5. If you win 30 points in a game, you might represent your score by $+30$. What would be meant by -15 ? by 25? by 0? by -50 ? by 100?



6. A gain, or profit, of \$50 might be represented by $+ 50$. How, then, would a gain of \$75 be represented? a loss of \$25? a loss of \$100? a gain of \$20?

7. Explain how improvement or the opposite might be indicated on your report card by the use of the signs $+$ and $-$.

8. If a balloon pulls upward with a force of 15 lb., how may this be represented? How would a weight attached to the balloon be represented? What would be the total effect if the weight were 1 lb.? 8 lb.? 12 lb.? $14\frac{1}{2}$ lb.? 15 lb.? 20 lb.?

9. If a boy can row in still water at the rate of 4 mi. an hour, how may this be represented? If he rows against the current of the river, how may the current be represented? What will be the effect if the current is $\frac{1}{2}$ mi. an hour? 2 mi. an hour? 3 mi.? 4 mi.? 5 mi.?

10. How may positive and negative numbers be used to indicate that one pipe is emptying water into a reservoir at the rate of 15,000 gal. a day and another taking it out at the rate of 8500 gal. a day?

11. If longitude east of Greenwich is positive, how could you locate places whose longitude is 5° ? -2° ? 18° ? 0° ? -20° ?

12. Explain a way in which north and south latitude may be indicated. How could you locate places whose latitude is $+22^\circ$? $+41^\circ$? 0° ? -10° ? -50° ?

13. In what way may a tug of war be represented by positive and negative numbers?

14. Describe other ways in which positive and negative numbers may be useful to designate opposite groups.

15. Which of the following numbers has the greatest value: 3, -8 , 15, 18, -20 , -25 ? Which has the least value? the least absolute value? the greatest absolute value?

16. Positive and negative numbers are sometimes called **signed numbers** or **directed numbers**. Explain why.

Adding Positive and Negative Numbers

(1) Suppose that you are playing a game and your score is 7 points to the good, or $+7$ points. If you gain 9 points in the next round, what will your score be then?

$$+7 \text{ points} + (+9 \text{ points}) = +16 \text{ points}$$

(2) If instead of winning 9 points in the second round, you lose 9 points, what will your score be then?

$$+7 \text{ points} + (-9 \text{ points}) = -2 \text{ points}$$

(3) If your original score is 7 points in the hole, or -7 points, and you gain 9 points, what will your score be then?

$$-7 \text{ points} + (+9 \text{ points}) = +2 \text{ points}$$

(4) If your original score is -7 points and you lose 9 points, what will your score be then?

$$-7 \text{ points} + (-9 \text{ points}) = -16 \text{ points}$$

EXERCISES

Translate into algebraic statements and find the sums:

1. John has \$14 and earns \$5.

2. After earning the \$5, John spends \$8 for a pair of shoes.

3. Mr. Jones borrows \$17; then receives \$42.50.

4. Fred owes \$12 and earns \$9 during the week.

5. Mary owes \$2.50 at a book store. She buys a book for \$1.50 and has it charged.

6. A man drives his car 12 mi. in an easterly direction and then 23 mi. in the opposite direction.

7. An aviator rises 5280 ft., descends 200 ft., rises 500 ft., then descends 1000 ft.

8. Alfred, who weighed 98 lb., became ill and lost 7 lb. in weight. After he recovered he gained 12 lb.

9. Joseph made 25 points in a game, gained 30 points, lost 50 points, lost 25 points, and then gained 15 points.

The exercises on page 27 explain the following rules for adding two algebraic numbers:

If two algebraic numbers have the same sign, find the sum of their absolute values and prefix their common sign.

If two algebraic numbers have unlike signs, find the difference of their absolute values and prefix the sign of the greater.

ORAL EXERCISES

Find the sums indicated:

$$\begin{array}{r} 1. \quad + 8 \\ \quad + 5 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad - 8 \\ \quad + 5 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad + 7 \\ \quad + 9 \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad - 7 \\ \quad + 9 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad + 10 \\ \quad - 12 \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad + 8 \\ \quad - 5 \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad - 8 \\ \quad - 5 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad + 7 \\ \quad - 9 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad - 7 \\ \quad - 9 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad - 18 \\ \quad + 32 \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad + 8 \\ \quad - 9 \\ \quad + 7 \\ \quad - 13 \\ \hline \end{array}$$

SOLUTION.

Add the positive numbers: $+ 8 + 7 = 15$

Add the negative numbers: $- 9 - 13 = - 22$

Find the sum of the results: $+ 15 + (- 22) = - 7$

Summary of Steps.

1. Find the sum of all the positive numbers.
2. Find the sum of all the negative numbers.
3. Find the sum of (1) and (2).

Add:

12.	15	13.	- 9	14.	7.5	15.	- 38.7	16.	$17\frac{1}{2}$
	12		5		- 9.3		- 52.5		19
	- 7		15.5		8.2		- 7.8		- 20
	- 3		- 8		1.6		85.2		- 45

17. Show that $5 \times 10 + 7 \times 10 + 3 \times 10 - 8 \times 10$ equals 7×10 .

18. Find the sum of $5t + 7t + 3t - 8t$.

Add:

$$19. 3 \times 4 + 5 \times 4 - 19 \times 4 + 13 \times 4 - 3 \times 4$$

$$20. 3f + 5f - 19f + 13f - 3f$$

21.	x	23.	$11a$	25.	$12p$	27.	$6r$
	$-17x$		$12a$		$-16p$		$12r$
	$13x$		$-15a$		$-15p$		$15r$
	$-20x$		$-a$		$8p$		$-25r$
	<u>$3x$</u>		<u>$-7a$</u>		<u>$23p$</u>		<u>$-7r$</u>

22.	$2a$	24.	$3b$	26.	$7x$	28.	$3n$
	$-3a$		$5b$		$-8x$		$2n$
	$5a$		$-8b$		$-2x$		$-n$
	$-7a$		$-2b$		$3x$		$-4n$
	<u>$9a$</u>		<u>$4b$</u>		<u>x</u>		<u>n</u>

Monomials and polynomials. When an algebraic expression is made up of parts connected by the signs $+$ and $-$, these parts are called **terms**.

Thus, in the expression $a + 2b + 3c$, the terms are a , $2b$, and $3c$. In $a + \frac{bx}{y} - c(a + b)$, the terms are a , $\frac{bx}{y}$, and $c(a + b)$. The expression $a + b$ consists of two terms; but when these numbers are grouped together by a parenthesis or by a bar in division, as $\frac{a + b}{c}$, they are considered as one number and the entire expression is a single term.

You may remember that the prefix *mono* (as in *monoplane*) means *one*; *bi* (as in *biplane*) means *two*; *tri* (as in *triangle*) means *three*; *poly* (as in *polygon*) means *many*; the root *nom* (as in *denominator*) means *name* or *term*.

An expression that consists of one term is called a **monomial**. (Why?) Thus, ab , $\frac{a - b}{c}$, and a^2bc are monomials.

An expression that consists of two or more terms is called a **polynomial**. (Why?) Thus, $a + b$, $2a - b + c$, and $\frac{1}{2}a + b - c + 3d$ are polynomials.

A polynomial that consists of two terms is called a **binomial**. (Why?) Thus, $a + b$, $c + ax^2y$, and $\frac{a+b}{c} - dx$ are binomials.

A polynomial that consists of three terms is called a **trinomial**. (Why?) Thus, $a + b + c$ and $\frac{a+b}{c} + d(x+y) + z$ are trinomials.

Terms that have the same literal factors with the same exponents, as $2ax$ and $7ax$, are called **like terms**, or **similar terms**.

Terms that do not have the same literal factors, as $4a$, $2b$, and $4ax$, are called **unlike terms**, or **dissimilar terms**.

EXERCISES

1. Select the monomials, the binomials, and the trinomials:

$b + c$	$x(b - c)$	$a^2 + bc - c^2$
bcx	$2a + b - c$	$\frac{n+p}{q} - n$

2. Select like terms:

bx	$2abx$	$5a^2x$	$8bx$
abx	$3bx^2$	$2b^2x$	$7b^2x$

3. How many different sets of unlike terms are there in ex. 2? Name each set.

Order of operations. Suppose that you are asked to find the value of $2 \times 4 + 3 \times 4 \div 3$. There are a number of ways in which you might get a result to this example; but in order to avoid confusion it has been decided to follow just one order, as shown below:

$$(2 \times 4) + (3 \times 4) \div 3 = 8 + \frac{12}{3} = 8 + 4 = 12$$

The advantage of following this order will be clear to you if you find the value of $ac + \frac{bc}{d}$ when $a = 2$, $b = 3$, $c = 4$, and $d = 3$.

EXAMPLE. What is the value of $3 \times 4^2 - 2 \times 4 \div 2 + 2^3$?

SOLUTION. $3 \times (4^2) - \frac{2 \times 4}{2} + 2^3 = (3 \times 16) - 4 + 8 = 52$

Compare this result with the value of the expression $ac^2 - \frac{bc}{2} + b^3$ when $a = 3$, $b = 2$, and $c = 4$.

Perform the operations in the following order:

1. Find all the indicated powers and roots.
2. Find all the indicated products and quotients.
3. Find all the indicated sums or differences.

Note that operations indicated within a parenthesis must be performed first of all. (See page 8.)

EXERCISES

Find the value of:

1. $18 \div 3 + 6 \times 7 \div 2$
2. $25 - 7 \times 3 + 18 \div 9$
3. $24 \times 3 \div 9 - 18 \div 6 + 3$
4. $10 + 8 \times 6 \div 12 - 9 \times 5 \div 15$
5. $15 + 5 \times 3 - 12 \times 9 \div 18 - 9$
6. $(15 + 5) \times 3 - 12 \times 9 \div (18 - 9)$
7. $18 - 3 \times 9 + 6 \div 3 + 12 - 5$
8. $(18 - 3) \times (9 + 6) \div (3 + 12) - 5$

Adding like terms. You have already noticed (page 28) that $5t + 7t + 3t - 8t = 7t$. In the same way, $5ax + 7ax + 3ax - 8ax = 7ax$.

EXAMPLE. Add $2abc - 3abc + 5abc + 7abc - 8abc$.

SOLUTION.

The sum of the positive terms is $2abc + 5abc + 7abc$, or $14abc$

The sum of the negative terms is $-3abc - 8abc$, or $-11abc$

The sum of $14abc$ and $-11abc$ is $3abc$

In adding like terms, find first the sum of the positive terms, then the sum of the negative terms. Combine these two sums and prefix the sign of the numerically greater of the two terms.

EXERCISES

Find the indicated sums :

1. $9p - 4p - 17p + 3p$
2. $15a + 3a - 9a + a - 8a$
3. $2xy - 3xy + 8xy - xy$
4. $5x^2y + 8x^2y - 7x^2y - 9x^2y$
5. $a^2b^2 - 9a^2b^2 + 2a^2b^2 + 4a^2b^2$
6. $2abc - 7abc + abc + 4abc$
7. $3x^2 - 7x^2 + 8x^2 + 5x^2 - x^2$
8. $9pq - 4pq - 17pq + 3pq$
9. $3r^2 + 5r^2 - 15r^2 + 6r^2$
10. $7(a + b) + 9(a + b) - (a + b)$
11. $3px + 8px - 2px - 7px$
12. $5(x + y) + (x + y) - 8(x + y) + 3(x + y)$
13. $2a^3 + 15a^3 - 7a^3 - 3a^3 + 4a^3$
14. $8a + 7a - 2a - 9a$
15. $8 \times 15 + 7 \times 15 - 2 \times 15 - 9 \times 15$
16. $13 \times 18 + 20 \times 18 - 8 \times 18 - 5 \times 18$
17. $15 \times 13 - 10 \times 13 - 17 \times 13 + 14 \times 13$

Adding unlike terms. Terms that are alike, as $5a$ and $7a$, can be added and their sum can be simply expressed, as $12a$; but there is no simple way of expressing the sum of terms that are not alike.

NOTE. If we add $5a$ and $8b$, the sum is neither $13a$ nor $13b$ nor $13ab$. This may be verified by substituting any values for a and b . Thus, if we substitute 2 for a and 3 for b , $5a + 8b = 5(2) + 8(3) = 34$. Also $13a = 26$, $13b = 39$, and $13ab = 78$. The only combination that gives the correct result is $5a + 8b$.

In adding a series of unlike terms, find the sums of the different sets of like terms and connect these sums with the correct sign, + or -.

EXAMPLE. Find the sum of $8a - 2b + 3a - c + b$.

SOLUTION. $8a + 3a = 11a$ and $-2b + b = -b$.

The sum is $11a - b - c$.

EXERCISES

Find the sum of:

- | | |
|--------------------------------|--|
| 1. $2a, 3b, -2a, -b$ | 6. $4r^2, -5r, 6, 8r, -3r^2, -2$ |
| 2. $5x, y, -x, 7x$ | 7. $3x^2, 6, -5x, -x^2, 8x$ |
| 3. $3a^2, 2ab, -3ab, 5a^2$ | 8. $3 \times 2, -5 \times 3, 8 \times 2, 2 \times 3$ |
| 4. $7x^2, -3y^2, 8x^2, -12x^2$ | 9. $7.5x, -5.2y, -3.3y, 2.3x$ |
| 5. $x, 2y, -y, 8x, -5y$ | 10. $5x^a, 2x^a, -y^b, -3x^a, 2y^b$ |

Adding Polynomials

EXAMPLE. Add $2a^2 + 3ab - b^2$, $-a^2 + 2b^2$, and $-5ab + b^2$. Check by letting $a = 2$ and $b = 3$.

SOLUTION.

$$\begin{array}{r}
 2a^2 + 3ab - b^2 \\
 -a^2 \qquad + 2b^2 \\
 -5ab + b^2 \\
 \hline
 a^2 - 2ab + 2b^2
 \end{array}$$

CHECK.

$$\begin{array}{r}
 = 8 + 18 - 9 = 17 \\
 = -4 \qquad + 18 = 14 \\
 = -30 + 9 = -21 \\
 = 4 - 12 + 18 = 10
 \end{array}$$

Polynomials are added by finding the sums of the like terms and arranging these sums as a new polynomial.

Checking. The accuracy of work in addition may be checked by reversing or otherwise changing the order of the work, since the order in which numbers are added does not change the result. Thus, $4 + 7 + 9 = 7 + 4 + 9 = 9 + 7 + 4 = 20$.

A common way to check algebraic addition is to substitute for the letters some conveniently small values, other than zero. If the numerical value of the sum is equal to the sum of the numerical values obtained by substitution in the given expressions, then the work is presumably correct. This is not an absolute check, for it is possible to make two mistakes that balance each other. If the two numerical results are not the same, there is a mistake either in the working of the problem or in the checking, and the work should be done over.

If $x = 1$, then $5x = 5$, $5x^2 = 5$, and $5x^3 = 5$. For this reason it is better not to use 1 as a value for checking. A convenient method to follow is to let the first alphabetical letter = 2, the second letter = 3, the third letter = 2, and so on. Thus, in checking $3x + 2y - z$, let $x = 2$, $y = 3$, and $z = 2$.

EXERCISES

Add, and check the results :

$$\begin{array}{r} 1. \quad 2a + 3b \\ \quad 5a + \quad b \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad 3x + 7y \\ \quad -x + 2y \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad -3p + 12q \\ \quad 8p - \quad 7q \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 5n - 3r \\ \quad -3n + 5r \\ \quad 2n + \quad r \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad -15xy + 8yz \\ \quad -2xy - 7yz \\ \quad 12xy + \quad yz \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 2x + 3y - 7z \\ \quad x - 2y + 8z \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 5a^2 - 2ab + 3b^2 \\ \quad -7a^2 + \quad ab - 4b^2 \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 12x^2 - 7x + 3 \\ \quad 2x^2 + 5x - 7 \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad 7a^2 - 2a + 8 \\ \quad -5a^2 + 3a - 2 \\ \quad -3a^2 - 5a - 7 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 2x^2 - 17xy + \quad y^2 \\ \quad -3x^2 + \quad xy + 5y^2 \\ \quad 8x^2 + \quad 3xy - 6y^2 \\ \hline \end{array}$$

Find the sums of the following polynomials :

11. $2a + 3b - c$, $3a + 2c$, $5b - 4c$, and $-4a + 2b + c$

12. $2x^2 - 5x + 3$, $4x^2 + x - 1$, $-5x^2 + 2x - 7$, and $7x + 5$.

13. $2x^2 + 3xy$, $7xy - y^2$, $4x^2 + 5xy - 8y^2$, and $-12x^2 + xy + 3y^2$

14. $2x + 3xy + 5$, $-3x - 2xy$, $7xy + 12 - 13x$, and $8x - 8$

15. $1.5a^2 - 2ab + 3b^2$, $2.3b^2 + 3ab - 7.1a^2$, and $2a^2 - ab + 2.2b^2$

16. $5a^3 + 3a^2 - 7a + 1$, $4a^2 + 2a - 8$, $-a^3 - 5a^2 + 3$, and $a^2 - a - 2$

17. $2p + 3q - 7s$, $5q + 3p + 2s$, $6s - 4p - 8q$, and $s + p - q$

18. $2a^2 + 3ab - 7b^2$, $-2ab + 7a^2 + 3b^2$, $5a^2 + 6ab + b^2$, and $-8a^2 + 12ab - 2b^2$

19. $4x + 3y - 7z$, $2x - 8y + z$, $9y - 7x - 10z$, and $17z + 3x - 2y$

20. $2x^3 - 3y^2 + 5$, $3x^3 + 2y^2 - 3$, $4x^3 - 7y^2 + 10$, and $-6x^3 + 5y^2 - 8$

21. $5xy + 3xz - 7yz$, $2xy - 10xz + yz$, $-8xy + xz - 2yz$, and $xy + 2xz + 4yz$

22. $8p^2 - pq + 3q^2$, $5p^2 + 7pq - 2.5q^2$, $2.3p^2 - 1.5pq + q^2$, and $-12p^2 + 2pq - 1.2q^2$

23. $8m - 2n + r$, $3m + 5n - 2r$, $3n - 7m + 3r$, and $m - 4n + 2r$

24. $.2a^2 + 1.5a - 3.5$, $.3a^2 - 2.5a + 4.5$, and $1.5a^2 + a - 2$

25. $3x^2 + 7xy + y^2$, $8xy - z^2 + yz$, $x^2 - 2y^2 + 3z^2$, and $3y^2 - 7yz$

26. $5x^3 + 3x^2y - 2y^2$, $4x^3 - x^2y - y^2$, $-3x^3 - 5x^2y + 8y^2$, and $x^3 - 2y^2$

27. $\frac{1}{2}a^2 - \frac{1}{3}ab + \frac{1}{4}b^2$, $a^2 + \frac{1}{2}ab - \frac{1}{2}b^2$, $2a^2 + ab + b^2$, and $\frac{1}{4}a^2 - \frac{2}{3}$

Complete these statements:

28. The distinguishing characteristic of positive and negative numbers is —.

29. The zero point of a thermometer scale indicates —.

30. The terms $5ax$ and $-2ax$ are like terms because —.

31. The expression $2x^2y$ is a monomial because —.

32. The expression $a^2 + b^2$ is a binomial because —.

33. The expression $x^2 + y^2 - z^2$ is a trinomial because —.

34. An expression containing more than one term is called a polynomial because —.

35. It is necessary to check all work because —.

Subtracting Algebraic Numbers

In finding the sum, s , of two numbers, a and b , one number is found that takes the place of the two numbers; as, $a + b = s$. A problem in subtraction is the opposite or inverse of one in addition. The sum of two numbers, s , and one of the numbers, a , are given and it is required to find the other number, b . The sum in this case is called the **minuend**, the given number is called the **subtrahend**, and the required number, or difference, is called the **remainder**.

Exercises in subtraction are frequently checked by finding the sum of the remainder and the subtrahend. If this sum equals the minuend, the result is correct. The exercise in subtraction then may be worded as an exercise in addition and the question asked may be, "What number added to the subtrahend will give the minuend?"

If Agnes spends 20¢ at the grocery store and hands the clerk a quarter, he will return to her 5¢ in change and will say "20 and 5 are 25."

NOTE. In this text, when the difference between two numbers is mentioned, the minuend is stated first. Thus, the difference between a and b means $a - b$.

ORAL EXERCISES

1. If you are playing a game and have a score of 32 points, how many points must you gain to have 50 points?

How many points must be added to your score of 32 points to give you 50 points?

Evidently the difference between 50 and 32 is 18, or $50 - 32 = 18$.

2. Suppose that your first score is 32 points and your second score is 15 points, how many points have you lost? What number added to 32 will give 15?

Evidently the difference between 15 and 32 is -17 , or $15 - 32 = -17$.

3. Suppose that your first score is -15 and your second score is -25 , how many points have you dropped back?

What number added to -15 will give -25 ?

Evidently the difference between -25 and -15 is -10 , or $-25 - (-15) = -10$.

4. Suppose that your first score is 10 and your second score is -15 , how many points have you fallen back?

What number added to 10 will give -15 ?

Evidently the difference between -15 and 10 is -25 , or $-15 - (+10) = -25$.

5. If a thermometer registered -4° at midnight and 8° the next noon, how many degrees did the temperature rise?

What is the difference between 8 and -4 ?

6. If a thermometer registered 10° at 2 P.M. and -5° at 1 A.M. the next night, how many degrees did the temperature fall?

Express this as a problem in subtraction.

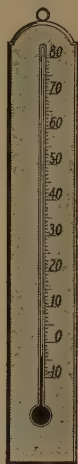
7. How many degrees change are registered when the temperature changes from -2° to -8° ?

8. What change is registered by a thermometer reading 65° at 8 A.M. and 78° at noon?

9. Find the difference in longitude between a place at 23° east longitude and another place at (a) 55° east longitude; (b) 5° east longitude; (c) 0° longitude; (d) 17° west longitude.

10. What is the difference in longitude between a place at 25° west longitude and a place at (a) 37° west longitude? (b) 10° west longitude? (c) 0° longitude? (d) 30° east longitude?

11. What change is registered by a thermometer reading -10° at 5 A.M. and 10° at noon? by a thermometer reading -5° at 6 A.M. and 15° at 3 P.M.?



ORAL EXERCISES

The first ten exercises just worked may be restated in algebraic numbers as follows. In each case, subtract the lower number from the upper one:

1. $\begin{array}{r} 50 \\ 32 \\ \hline \end{array}$	5. $\begin{array}{r} 8 \\ - 4 \\ \hline \end{array}$	9. (a) $\begin{array}{r} 23 \\ 55 \\ \hline \end{array}$	10. (a) $\begin{array}{r} - 25 \\ - 37 \\ \hline \end{array}$
2. $\begin{array}{r} 15 \\ 32 \\ \hline \end{array}$	6. $\begin{array}{r} - 5 \\ 10 \\ \hline \end{array}$	(b) $\begin{array}{r} 23 \\ 5 \\ \hline \end{array}$	(b) $\begin{array}{r} - 25 \\ - 10 \\ \hline \end{array}$
3. $\begin{array}{r} - 25 \\ - 15 \\ \hline \end{array}$	7. $\begin{array}{r} - 8 \\ - 2 \\ \hline \end{array}$	(c) $\begin{array}{r} 23 \\ 0 \\ \hline \end{array}$	(c) $\begin{array}{r} - 25 \\ 0 \\ \hline \end{array}$
4. $\begin{array}{r} - 15 \\ 10 \\ \hline \end{array}$	8. $\begin{array}{r} 78 \\ 65 \\ \hline \end{array}$	(d) $\begin{array}{r} 23 \\ - 17 \\ \hline \end{array}$	(d) $\begin{array}{r} - 25 \\ 30 \\ \hline \end{array}$

Subtracting like terms. You have seen that a problem in subtraction may be stated so that it becomes a problem in addition. Thus, asking for the difference between 50 and 32 is the same as asking, "What number added to 32 will give 50?"

$$\begin{array}{r} \text{Subtract: } 50 \\ 32 \\ \hline 18 \end{array}$$

$$\begin{array}{r} \text{CHECK. Add: } 32 \\ 18 \\ \hline 50 \end{array}$$

The result is obtained by adding 50 and $- 32$.

Similarly, the difference between 8 and $- 4$ may be found by finding the number that added to $- 4$ will give 8.

$$\begin{array}{r} \text{Subtract: } 8 \\ - 4 \\ \hline 12 \end{array}$$

$$\begin{array}{r} \text{CHECK. Add: } - 4 \\ 12 \\ \hline 8 \end{array}$$

This result is obtained by adding 8 and 4.

To find the difference between two numbers, change the sign of the subtrahend and find the algebraic sum of the subtrahend and the minuend.

EXERCISES

Subtract the lower number from the upper one:

$$\begin{array}{llll} 1. & 9a & 5. & 15x^2 \\ & \underline{5a} & & \underline{17x^2} \end{array} \quad \begin{array}{llll} 9. & -12ab & 13. & 0 \\ & \underline{3ab} & & \underline{-15x} \end{array}$$

$$\begin{array}{llll} 2. & 3a & 6. & -25a^3 \\ & \underline{5a} & & \underline{5a^3} \end{array} \quad \begin{array}{llll} 10. & -5ax & 14. & 3.2pq \\ & \underline{-11ax} & & \underline{4.9pq} \end{array}$$

$$\begin{array}{llll} 3. & 8a & 7. & 0 \\ & \underline{-2a} & & \underline{17a} \end{array} \quad \begin{array}{llll} 11. & -8a^2y & 15. & 1.8rs^2 \\ & \underline{17a^2y} & & \underline{-.5rs^2} \end{array}$$

$$\begin{array}{llll} 4. & 7a & 8. & 0 \\ & \underline{-9a} & & \underline{-8p} \end{array} \quad \begin{array}{llll} 12. & 32xy^2 & 16. & -7.3m^2n \\ & \underline{40xy^2} & & \underline{-8.1m^2n} \end{array}$$

$$\begin{array}{llll} 17. & 3ax & 19. & 2\frac{1}{3}ax \\ & \underline{-7ax} & & \underline{3\frac{1}{2}ax} \end{array} \quad \begin{array}{llll} 21. & 3(a+b)^2 & & \\ & \underline{-2(a+b)^2} & & \end{array}$$

$$\begin{array}{llll} 18. & a^2b & 20. & 4\frac{1}{5}b^2y \\ & \underline{19a^2b} & & \underline{-3\frac{2}{5}b^2y} \end{array} \quad \begin{array}{llll} 22. & -6a(x^2+y) & & \\ & \underline{-9a(x^2+y)} & & \end{array}$$

$$\begin{array}{llll} 23. & 0 & 24. & 13(a+b) \\ & \underline{-7bxy} & & \underline{20(a+b)} \end{array}$$

25. What number added to $-18a$ will give $27a$?26. Find the number that added to $23xy$ will give 0.27. From the sum of $15x^2y$, $2x^2y$, $-8x^2y$, and x^2y subtract $-5x^2y$.28. Subtract the sum of $5b(x+y)$, $-7b(x+y)$, and $-3b(x+y)$ from the sum of $2b(x+y)$, $-8b(x+y)$, and $b(x+y)$.29. What number added to the sum of 81, -71 , and 151 will give 1?30. What number subtracted from $5a$ will give $3a$?31. What number subtracted from $-10a$ will give $7a$?

Subtracting Polynomials

Before trying to subtract one polynomial from another polynomial, place the subtrahend below the minuend so that like terms are in the same columns.

EXAMPLE. Subtract $2a + 7b - 3c$ from $5a - b + 2c$.
Check.

SOLUTION.	CHECK. (1) $a = 2, b = 3, c = 2$	(2) By addition
$5a - b + 2c$	$= 10 - 3 + 4 = 11$	$2a + 7b - 3c$
$2a + 7b - 3c$	$= 4 + 21 - 6 = 19$	$3a - 8b + 5c$
$3a - 8b + 5c$	$= 6 - 24 + 10 = -8$	$5a - b + 2c$

The checking may be done by substitution, as in (1), or by finding the sum of the subtrahend and the remainder, as in (2). One method of checking is sufficient.

EXERCISES

Subtract and check:

$$\begin{array}{r} 1. \quad 3a - 2b \\ \quad a + b \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 3a + 2b - 3c \\ \quad a - b \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad 5a + 3b \\ \quad 6a - 4b \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad 5x^2 - 6xy + y^2 \\ \quad x^2 - 2xy + 3y^2 \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad -3n + 7m \\ \quad 5n - 4m \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 2a^2 - 3ab + 2b^2 \\ \quad 3a^2 + ab - b^2 \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 8p + q \\ \quad 6p - q \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad r^2 + rs \\ \quad 3r^2 - rs + t^2 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad 6ax - 7by \\ \quad 4ax + 9by \\ \hline \end{array}$$

$$\begin{array}{r} 12. \quad 0 \\ \quad -x^2 + xy - 2y^2 \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad 13xy - 5 \\ \quad -xy + 10 \\ \hline \end{array}$$

$$\begin{array}{r} 13. \quad 17p^2 - pq + 9q^2 \\ \quad 3p^2 + 8pq + 9q^2 \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad 8abc \\ \quad -5abc + 4 \\ \hline \end{array}$$

$$\begin{array}{r} 14. \quad 2.3m - 3.5n - 4.0p \\ \quad 1.5m + 2.2n + .3p \\ \hline \end{array}$$

15. Subtract $2x^2 - 7xy + 3y^2$ from $3x^2 + 2xy - y^2$.
16. Subtract $3xy + 7x^2 - y^2$ from $5x^2 - xy + 7y^2$.
17. Subtract $2a + b$ from $3b + 2c$.
18. Subtract $a + 2b + 5$ from $a^2 + 3a - b$.
19. Subtract $3x - y + 2z$ from $x + 5y - 7z$.
20. From $5r - 2s + t$ subtract $r + 8s - 3t$.
21. From 0 subtract $5a^2 - 2ab + 9b^2$.
22. From the sum of $2a + 3b - c$ and $-a + 5b + 2c$ subtract $7b - a + 3c$.
23. From the sum of $2x - 7y$ and $8y + z$ subtract $x - y - z$.
24. What expression added to $7x + 3y - 8z$ will give $x + y + z$? Check with $x = 2$, $y = 3$, and $z = 2$.
25. What expression added to $8a^2 - 2ab + b^2$ will give $9a^2 - 6ab + b^2$?
26. What expression subtracted from $5x^2 - xy + 7y^2$ will give $10x^2 - 2xy + 15y^2$? Check with $x = 2$, $y = 3$.
27. What expression subtracted from $x^2 - (a + b)x + 5$ will give $2x^2 + 3(a + b)x - 1$?
28. How much larger than 8 is 12? How much larger than $2a^2 + 3ab - b^2$ is $7a^2 - ab + 2b^2$? Check.
29. How much smaller than 15 is 8? How much smaller than $5x^2 + xy - 3y^2$ is $2x^2 - 7xy - y^2$? Check.
30. By how much does 12 exceed 5? By how much does $3x^2 + 2x - 7$ exceed $x^2 - 7x + 1$? Check.
31. Find the difference between $5a^2 - 2ab + b^2$ and $a^2 + 3ab$. Check.
32. What number added to $8x + 7y - z$ will give $3x + y + 5z$?
33. From the sum of $3a + 5b - 6c$ and $8b - 5a + 5c$ subtract $7c + a - 10b$. Check.

REVIEW EXERCISES

1. How can you write the sum of $15x$ and $8x$? the sum of $15x$ and $8y$?

2. Define "algebraic expression." Give illustrations.

3. What is a term? a monomial? a binomial? a trinomial? a polynomial?

4. Classify each of the following expressions:

(1) $r^2 + 2rs + s^2$

(3) $a(x + y)$

(5) $(2a - b)c$

(2) $3x - y^2$

(4) $a + b^2c$

(6) $a + \frac{bc - c^2}{a}$

5. Distinguish between like terms and unlike terms. Write three of each.

6. Define "positive numbers"; "negative numbers." Give practical examples of positive and negative numbers.

7. What is the absolute value of a number? Which has the greatest absolute value -2 , 3 , 9 , or -15 ? the least absolute value? the greatest actual value? the least?

8. Which is larger, 3^2 or 2^3 ? $(\frac{1}{2})^2$ or $(\frac{1}{3})^2$? 1^2 or 1^3 ?

9. Which is larger, n or n^2 , when n is greater than 1 ? when n is less than 1 ?

10. What is the value of: (a) 2×2^3 ? (b) 2×3^2 ? (c) $2^2 \times 2^3$? (d) 3×3^2 ?

11. Write the formula for the perimeter, p , of a triangle whose sides are a , b , and c . What kind of expression is the right side of the formula?

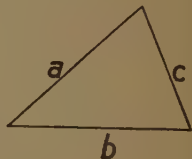
12. In the triangle at the right:

(a) Find p if $a = 8$, $b = 9$, $c = 12$.

(b) Find a if $p = 20$, $b = 7$, $c = 8$.

(c) Find b if $p = 38$, $a = 12$, $c = 15$.

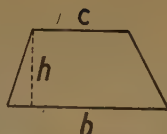
(d) Find c if $p = 28$, $a = 9$, $b = 13$.



13. What is the formula for the circumference of a circle? Find C when $r = 18.2$. Find r when $C = 96.9$.

14. What is the formula for the area of a trapezoid? Find A when $c = 8$, $b = 10$, $h = 5$.

15. Find h of a trapezoid when $A = 66$, $b = 12$, and $c = 10$.



16. Write a formula for the total area of the six faces of a rectangular solid if l , w , and h represent the length, the width, and the height of the solid.

17. Find the total surface of a rectangular solid if $l = 5$, $w = 4$, and $h = 12$.



18. Why is it advisable to check the result obtained in an exercise? State two methods of checking an exercise in addition; an exercise in subtraction.

19. If n is any integer, show that $2n$ must be an even number and $2n + 1$ must be an odd number.

20. What are consecutive numbers? Write three consecutive numbers the largest of which equals n . Write three consecutive numbers the smallest of which equals a .

Find the value of:

21. $6 + 9 \times 4 + 12 - (8 + 7) + 3 - 2$

22. $(6 + 9) \times 4 + 12 - 8 + 7 + (3 - 2)$

23. $(6 + 9) \times (4 + 12 - 8) + 7 + 3 - 2$

24. Find the sum of $15 \times 17 + 13 \times 17 - 8 \times 17 - 10 \times 17$.

25. Does $x^2 - 7x + 12$ equal 0 if $x = 1$? if $x = 3$? if $x = 4$? if $x = 2$?

26. Does $2x^2 - 14x$ equal 16 if $x = -1$? if $x = 3$? if $x = 4$? if $x = 5$? if $x = 8$?

27. If $C = \frac{5}{9}(F - 32)$, find the value of C when $F = 0^\circ$; when $F = -13^\circ$; when $F = 68^\circ$.

28. Write each of the following terms without the use of exponents:

(a) $5x^2$

(b) $3ay^3$

(c) x^2y

(d) $2a^4x$



Fun with Figures



1. Find the sum of the numbers in each horizontal row, in each vertical row, and in each diagonal row of the square at the right.

2. (a) What is the sum of the numbers 3, 12, 14, and 5? 6, 7, 11, and 10? Can you discover any other similar arrangements of four numbers whose sum is 34?

(b) Find the sum of the four numbers in the corners; of the four numbers in the center; of the pairs of middle numbers on opposite sides.

16	2	11	5
3	13	8	10
6	12	1	15
9	7	14	4

FIG. 1

An arrangement of numbers similar to that given in Fig. 1 is called a **magic square**.

3. Write a magic square with the numbers 1 to 9 so arranged that the sum of any three numbers in a straight line is 15.

4. Find the sum of the numbers in each horizontal row, in each vertical row, and in each diagonal row of the magic square at the right.

5. What is the sum of the three middle numbers on each of the two opposite sides? of the numbers in the four corners of the outer square and the four corners of the inner square?

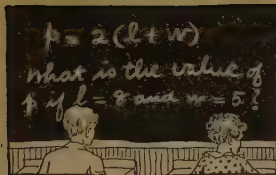
6. Show that the inner square of Fig. 2 is a magic square.

25	6	7	24	3
4	10	17	12	22
5	15	13	11	21
8	14	9	16	18
23	20	19	2	1

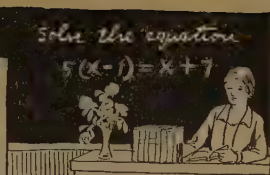
FIG. 2

7. Arrange the numbers 1 to 9 on the sides of a triangle so that the sum of the numbers in each straight line equals 17.

The subject of magic squares is fascinating. You will be able to find more about them by consulting such books as "Mathematical Recreations and Essays," by Ball; "A Scrapbook of Elementary Mathematics," by White; "Recreations in Mathematics," by Lick.



CHAPTER III



HOW TO USE PARENTHESES

If you wish to write a formula for the statement that the perimeter of a parallelogram equals twice the length plus twice the width, you will find it convenient to group the numbers representing the length and the width together in a parenthesis, thus:

$$p = 2(l + w)$$

You remember that the amount of a sum of money at interest may be expressed by the formula

$$A = p + prt$$

in which A represents the amount, p , the principal, r , the rate, and t , the time. Express the same formula using a parenthesis with the coefficient p .

Sometimes one parenthesis occurs within another. In such cases it is convenient to use other signs of grouping, as *brackets* $[\]$, *braces* $\{\}$, and the *bar* — . The bar is used almost exclusively to separate the numerator of the fraction from the denominator, as in the fraction $\frac{a + b}{c}$.

These four signs are called **signs of grouping**, or **signs of aggregation**. Where no confusion is possible, the term *parenthesis* is used as a common term for all these signs.

The expression within the parenthesis is considered as one number. (See page 8.)

Thus, $2(l + w)$ means that the number $l + w$ is to be multiplied by 2. If the parenthesis is removed, each of the numbers, l and w , is to be multiplied by 2. Hence $2(l + w) = 2l + 2w$.

ORAL EXERCISES

Read each exercise, remove the parenthesis, and simplify :

- | | | |
|----------------|-------------------|------------------|
| 1. $3(a + b)$ | 4. $3(a^2 - b^2)$ | 7. $4(3x - y)$ |
| 2. $2(b + c)$ | 5. $8(xy - 5)$ | 8. $2(7ax + 3)$ |
| 3. $2(2a - b)$ | 6. $2(2x + 3)$ | 9. $9(x^2 - xy)$ |

10. Find the value of each of the expressions in ex. 1 to 4 if $a = 3$, $b = 2$, $c = 3$.

Removing Parentheses

Suppose that you wish to find the sum of $2a$ and $a - b + c$, that is, the value of $2a + (a - b + c)$.

You have noticed that when two algebraic numbers are added, no signs are changed, but like terms are combined and others are written after them.

SOLUTION. Arrange the expressions as for addition :

$$\begin{array}{r} 2a \\ a - b + c \\ \hline \text{Add: } 3a - b + c \end{array}$$

This is equivalent to writing $2a + a - b + c$ and combining the like terms so as to get $3a - b + c$.

Suppose that you wish to subtract $2x + y - z$ from $3x$, that is, to find the value of $3x - (2x + y - z)$.

When a number is subtracted from another number, the signs of the subtrahend are changed and the subtrahend is added to the minuend.

SOLUTION. Arrange the expressions as in subtraction :

$$\begin{array}{r} 3x \\ 2x + y - z \\ \hline \text{Subtract: } x - y + z \end{array}$$

This is equivalent to combining like terms in the expression $3x - 2x - y + z$ so as to get $x - y + z$. The quantity within the parenthesis is to be subtracted, hence all the signs must be changed and the resulting expression must be added.

A parenthesis preceded by a plus sign may be removed without changing any signs within the parenthesis.

A parenthesis preceded by a minus sign may be removed if the sign of each term within the parenthesis is changed.

EXERCISES

Remove each parenthesis and combine like terms:

- | | |
|--|------------------------------|
| 1. $5 + (7 - 2)$ | 10. $3xy - (2xy + y^2)$ |
| 2. $8 + (9 + 1)$ | 11. $(2x - 7y) + 3x$ |
| 3. $4 - (5 - 2)$ | 12. $(-5a + 3b) + 7a$ |
| 4. $7a + (b - a)$ | 13. $8a - (-2a - b)$ |
| 5. $8x - (2x - y)$ | 14. $(a + b) - (5b - 3a)$ |
| 6. $x + (4x - z)$ | 15. $(2a - 5b) + (3a + 2b)$ |
| 7. $5a^2 + (b^2 - 2a^2)$ | 16. $6a + 5b - (2a - b + c)$ |
| 8. $7pq - (8 + pq)$ | 17. $(2a - b) + (2b - 5a)$ |
| 9. $r^2 + (s^2 - 5r^2)$ | 18. $(2n + 3x) - (-2n - 5y)$ |
| 19. $(3x - 2y - z) - (5x - 2y + z)$ | |
| 20. $a^2 - 2ab - (3ab + b^2)$ | |
| 21. $2m - n - (3n - 4m)$ | |
| 22. $(2a^2 - 3ab) - (-a^2 + ab - b^2)$ | |
| 23. $7a - (2a - 3b + 5c) + b + c$ | |
| 24. $3a - b - (-2a + 7b - 2c)$ | |
| 25. $(2a + b) - (3b + a) - (5a + c)$ | |
| 26. $(2a + b - c) - (-3b - c + a)$ | |

27. By means of a parenthesis, indicate the sum of the two expressions $3a - 2b + c$ and $5a + 3b - 2c$. Find the sum.

28. By means of a parenthesis, indicate that $3x - 2y + z$ is to be subtracted from $7x + 3y - 5z$. Find the difference.

29. Using three sets of parentheses, indicate that $2a - b + 5c$ is to be subtracted from the sum of $3a - 7c$ and $5a + 3b$.

30. Simplify $(3a - 2b + 5c) - (5b + 3c - 2a) - (4c + 3a - 3b)$ by removing the parentheses and combining like terms. Check the result with $a = 2$, $b = 3$, and $c = 2$.

Removing parentheses within parentheses.

EXAMPLE. Simplify $2a - [b - 3a - (2a + 3b)]$.

SOLUTION. Remove the inner parenthesis: $2a - [b - 3a - 2a - 3b]$

Remove the brackets: $2a - b + 3a + 2a + 3b$

Combine like terms: $7a + 2b$

In removing a series of parentheses within parentheses, remove the parentheses in succession, beginning with either the innermost or the outer ones. Continue till all of the parentheses have been removed. Combine like terms.

EXERCISES

Remove the parentheses and simplify:

1. $3 + (17 - 8) - [8 - (15 - 3)]$
2. $[7x - (2y + z)] - [3x + (2y - z)]$
3. $[a^2 + (b^2 - c^2)] - [(2a^2 - b^2) + 3c^2]$
4. $3x + 2y - [5x - (2y - z) + 3y] - 3z$
5. $[5a - (2b - c)] + [5c - (2a + b)]$
6. $7x^2 - (y - 2z) - [(3x^2 + y) - 3z]$
7. $(r^2 - s^2 + t^2) + [3t^2 - (2s^2 + r^2)]$
8. $(5x + 3y) - 2z - [(2x - 5y) - (7z + 3x) - 2y]$
9. $a^2 - \{2b^2 + [3a^2 - (b^2 - c^2) + b^2] - 3a^2\}$
10. $3x^2 + (2xy - y^2) + \{3y^2 - [5x^2 - (3xy - 7y^2)] - x^2\}$
11. $3a - [2b - (3c + 5a)] - (2a - c)$.
12. $7a - [-2b + (-3a - c + b)]$.

Grouping Terms within Parentheses

It is sometimes convenient to group terms together in a parenthesis. A parenthesis preceded by the sign $+$ may be removed without changing the signs of any of the terms within the parenthesis. Similarly, terms may be placed within such a parenthesis without any change of sign. Thus, $3 + 15 - 8 = 3 + (15 - 8)$, since each expression equals 10. Similarly, $a + b - c = a + (b - c)$.

In removing a parenthesis preceded by the sign $-$, the sign of each term within the parenthesis must be changed. Similarly, when terms are grouped within a parenthesis preceded by the sign $-$, the sign of each term placed within the parenthesis must be changed. Thus, $18 - 13 + 6 = 18 - (13 - 6)$, as each member equals 11. Similarly $a - b + c = a - (b - c)$.

Terms may be inclosed within a parenthesis preceded by a plus sign without change of any signs.

Terms may be inclosed within a parenthesis preceded by a minus sign provided the sign of each term within the parenthesis is changed.

EXERCISES

Group the last two terms of each of the following expressions in a parenthesis preceded by a plus sign :

1. $5 + 7 - 3$

5. $3ay + 2by - 2bx + 3ax$

2. $15 + 9 - 4$

6. $nrs - nr - ns$

3. $a + b - c$

7. $ab + ac - bx - cx$

4. $3x + y - z$

8. $m + n + mp + np$

Group the last two terms of each of the following expressions in a parenthesis preceded by a minus sign :

9. $a - b - c$

12. $x^2 - ax + bx$

10. $x^2 + cy - xz$

13. $a + b - ac - bc$

11. $2ax - bx - 2az + bz$

14. $mn - pn - mx + px$

Group the last three terms of each of the following expressions within a parenthesis :

15. $a^2 - b^2 - 2bc - c^2$

18. $9x^2 - 4a^2 + 4ab - b^2$

16. $x^2 - y^2 + 2yz - z^2$

19. $16a^2 - 9b^2 - 30bc - 25c^2$

17. $a^2 - x^2 - 2xy - y^2$

20. $x^2 - y^2 - 4a^2 + 4ab - b^2$

21. $r + s + ax + ay + az$

22. $2x^2 + 2xy - 25a^2 - 10ab - b^2$

23. $36 - 49m^2 + 14mn - n^2$

24. $49x^2 - a^2 - 12ab - 36b^2$

Literal coefficients. If you are asked for the sum of $5x$ and $3x$, you add the coefficients 5 and 3, getting 8, and prefix this to x , with this result: $5x + 3x = (5 + 3)x = 8x$. In a similar way, the sum of ax and bx is found by writing the sum of the coefficients a and b as one number prefixed to x , thus: $ax + bx = (a + b)x$.

EXERCISES

Combine the coefficients of x and n and write as one term:

$$1. \quad ax - bx \qquad \text{Ans. } (a - b)x$$

$$2. \quad an + bn - cn \qquad \text{Ans. } (a + b - c)n$$

$$3. \quad a^2x + b^2x$$

$$5. \quad cx - dx + ex$$

$$7. \quad 3an - 2bn - n$$

$$4. \quad mx + qx - px$$

$$6. \quad 2ax + 3bx + x$$

$$8. \quad rx - sx + tx$$

Add:

$$9. \quad lx$$

$$\underline{mx}$$

$$\text{Ans. } (l + m)x$$

$$10. \quad jy$$

$$\underline{-ky}$$

$$11. \quad ax + by$$

$$\underline{-bx - cy}$$

Subtract the lower number from the upper one:

$$12. \quad ap$$

$$\underline{p}$$

$$\text{Ans. } (a - 1)p$$

$$13. \quad abx$$

$$\underline{-cx}$$

$$14. \quad 2mx^2$$

$$\underline{nx^2}$$

$$15. \quad 3x - 2y + z$$

$$\underline{ax + by - cz}$$

$$16. \quad (a + b)x - (c + d)y$$

$$\underline{2bx + 3dy}$$

Unite the coefficients of x and y and write as a binomial:

$$17. \quad ax - by - bx + y - 2x$$

$$\begin{aligned} \text{SOLUTION. } ax - by - bx + y - 2x &= ax - bx - 2x - by + y \\ &= (a - b - 2)x - (b - 1)y \end{aligned}$$

$$18. \quad 2a^2x - 3abx - c^2y + x - cy$$

$$19. \quad 5x + 2ay - by + ax - cx + cy$$

$$20. \quad mx + my - nx - ny + x + y$$

$$21. \quad 2ax - 2ay - bx + by + 3cx - 3cy$$

Parentheses in equations.**EXAMPLE.** Solve $3(x + 2) - 5 = 8 - (x - 1)$.

SOLUTION. $3(x + 2) - 5 = 8 - (x - 1)$ (1)

Remove the parentheses in (1): $3x + 6 - 5 = 8 - x + 1$ (2)

Combine like terms in (2): $3x + 1 = 9 - x$ (3)

Add $x - 1$ to both sides of (3): $4x = 8$ (4)

Divide both sides of (4) by 4: $x = 2$ (5)

Check by substituting 2 for x in (1):

$3(2 + 2) - 5 = 8 - (2 - 1)$

$3 \times 4 - 5 = 8 - 1$

$7 = 7$

EXERCISES

Solve the following equations. Check ex. 1 to 10:

1. $16x + (12 - 3x) = 64$

8. $7x - 2(3x + 5) = 8$

2. $2(x + 1) = x + 5$

HINT. $7x - (6x + 10) = 8$

3. $14x - 5 = 3(2x + 5)$

9. $6x = 18 - 3(x + 1)$

4. $5n - 4 = 3(n + 2)$

10. $7(x - 2) - 5(x - 2) = 8$

5. $7n = 2(n + 1) + 13$

11. $5(2x - 1) - 2(3x + 2) = x$

6. $5x = 3 + 4(x + 1)$

12. $2(2n - 1) - n = 25$

7. $5x + 3 = 2(x + 6)$

13. $2(n - 5) - 3(4 - n) = 18$

14. $3x + 19 = 5x - 4(2x + 1)$

15. $3 + 7(2x + 1) = 3(3x + 5)$

16. $3(n + 3) + 2(n - 2) = 15$

17. $2(n - 5) - 3(4 - n) = 1$

18. $6n - 1 = 5n - 3(n - 5)$

19. $4 + 5x = 3(x + 10)$

20. $3n + 5(2n + 3) - 28 = 0$

21. $4x - 2 = 2(x + 1) + 1$

22. $5x + 4 = (x + 7) - (3x + 1)$

23. $2(n + 1) - 4(n + 3) = 3(2 - n)$

24. $3(n - 3) - (n - 1) - n = 2(n - 2)$

25. $4(n - 1) - 3(3n - 8) + 5 = n - 5$

26. $.5n + 2(n - 3) = .25n + 12$

27. $.3(n - .5) - .7n = 2.1 - .5(2n - 1.5)$

Algebraic Representation

EXERCISES

1. The sum of two numbers is 25. If one of them is n , what is the other number?

2. The difference between two numbers is d . If the smaller number is 8, what is the larger number?

3. Express the larger of two numbers when the smaller number is s and their difference is 15.

4. Divide 33 into two parts one of which is p .

5. By how much does n exceed 5?

6. By how much does 18 exceed $n - 7$?

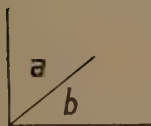
7. John is y years old. Express his age 5 yr. ago; 3 yr. hence.

8. Erwin is n years old. Express David's age if it is 4 yr. less than twice Erwin's age.

9. Express in cents the value of q quarters; of d dimes.

10. Express in dollars the value of h half dollars; of q quarters; of d dimes.

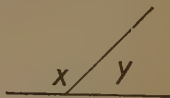
11. If there are 18 coins in a purse and n of them are nickels, how many other coins are there?



12. Two angles, a and b , are **complementary** if their sum is 90° . If one of two complementary angles is 40° , what is the other angle?

13. An angle is d° . Find its complementary angle.

14. Two angles, x and y , are **supplementary** if their sum is 180° . If one of two supplementary angles is 80° , what is the other angle?



15. If one angle is n° , what is its supplementary angle?

16. The sum of the angles of a triangle is 180° . Express the third angle if one angle is n° and the second angle is 25° less.

How to Attack a Problem

Before attempting to solve a problem, read it carefully to find out just what facts are given and what is to be found.

If only one number is to be found, represent it by some letter, as n for number, y for the number of years, l for the number of feet in the length, etc. If more than one number is required, select the number that seems to be the principal one and represent it by a letter. Frequently it is easiest to use the smallest number as the principal number.

From the conditions of the problem, find two algebraic expressions that are equal to each other.

Form the equation and solve it.

Check the result by substituting in the *original problem*. It is not sufficient to check the result in the *original equation*, since you may have made an error in stating the equation.

EXERCISES

1. The sum of three numbers is 55. The second number is 5 more than the first one and the third number is twice the second number. Find the three numbers.

SOLUTION.

Let n = the first number

Then $n + 5$ = the second number

and $2(n + 5)$ = the third number

$n + (n + 5) + 2(n + 5)$ = the sum of the numbers

Then $n + (n + 5) + 2(n + 5) = 55$

Remove the parentheses and combine like terms:

$$4n + 15 = 55$$

Solve:

$$\left. \begin{array}{l} n = 10 \\ n + 5 = 15 \\ 2(n + 5) = 30 \end{array} \right\} \text{Ans.}$$

CHECK. The sum of 10, 15, and 30 is 55. Also, 15 is 5 more than 10, while 30 is twice 15. Hence the numbers answer the conditions of the problem.

2. The sum of two numbers is 28. If 4 times the smaller number is equal to 3 times the larger one, find the numbers.

3. Find two numbers whose difference is 7 if twice the larger number is 6 more than 3 times the smaller one.

4. The smaller of two numbers is 7 and their difference is d . Three times the larger number is equal to 8 times their difference increased by 1. What is the larger number?

5. Find two numbers whose sum is 31 if twice one of the numbers diminished by 3 times the other number is 2.

6. Divide 36 into two parts such that the larger part increased by 6 equals twice the smaller part.

7. Find two consecutive integers if 3 times the first increased by twice the second is 137.

8. Find two consecutive integers if 3 times the larger integer decreased by twice the other one is 12.

9. Find three consecutive odd integers such that the first integer plus twice the second plus 3 times the third equals 46.

HINT. $n + 2(n + 2) + 3(n + 4) = 46$. Explain.

10. A 60-foot flagpole was broken by the wind into two parts such that the part broken off was 9 ft. longer than twice the part left standing. Find the length of each part.

Problems about angles and triangles.

11. Find two complementary angles whose difference is 25° .

HINT. $a - (90 - a) = 25$. Explain.

12. Find an angle that exceeds its complement by 18° .

13. How large is an angle if the difference between 3 times the angle and twice its complement is 20° ?

14. Find two supplementary angles if one of the angles exceeds the other by 34° .

15. What angle has the sum of its complement and its supplement equal to 200° ?

16. The sum of the angles of a triangle is 180° . If angle A is twice angle B and angle C is 5° larger than angle A , find the size of each of the angles of triangle ABC .

REVIEW EXERCISES

1. What are the signs of grouping?
2. Give an illustration in which such signs are helpful.

Remove the signs of grouping and simplify each of the following expressions by combining like terms:

3. $5(a - b) + 3(2b - 5a) - 2(b - 3a)$

HINT. $(5a - 5b) + (6b - 15a) - (2b - 6a) = ?$

4. $3(x - y) - 2(y - 2x) + 5(x - 3y)$

5. $7(p - 2q) + 3(3q - 2p) - 2(3p + q)$

Combine the coefficients of x into a single term:

6. $ax + bx - cx$

8. $abx - cx + x$

7. $5x - ax + bx$

9. $7ax + 5bx - bcx$

10. State the exponents and coefficients of x in the expression:

$$(2a - b)x^3 - (3b - c)x^2 + (a + b - c)x$$

11. How many terms are there in ex. 10? Read each term.

12. What expression added to $(3a + b)x$ gives $(5a - 3b)x$?

HINT. $(3a + b)x = 3ax + bx$ and $(5a - 3b)x = 5ax - 3bx$.

13. From the sum of $(a + 2b)x$ and $(3b - 2c)x$ subtract $(b - 2a + c)x$.

14. How many x 's are there in $5x + 8x$? in $ax - 2bx - 3cx$?

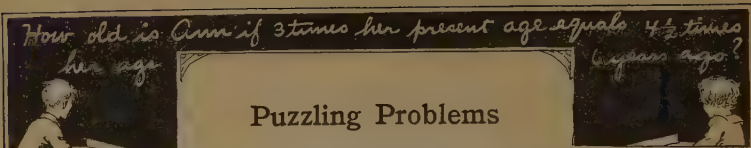
15. How many y 's are there in $3ay - 2by + y$?

16. How many parentheses of $(a + b)$ are there in the expression $(a + b)x - (a + b)y$?

17. How many $(x + y)$'s are there in the expression $2(x + y)a - 3(x + y)b$?

18. What is the area of a parallelogram if the sum of its base and its altitude is 39 in. and their difference is 9 in.?

19. Formulate a rule for removing in one operation all the signs of grouping in an expression.



Puzzling Problems

1. John and Arthur take positions at an initial salary of \$750 a year. John is to get an increase of \$45 every 6 mo. and Arthur is to get an annual increase of \$100. Which has the better paying position for the first 4 years? Why?

2. Mary accepted a position that paid \$30 a week. Agnes took one that paid \$5 the first week with an increase each succeeding week equal to the salary of the preceding week. Which girl had the better paying place if they remained 3 wk.?
5 wk.?

3. John sells $\frac{1}{5}$ of his goods for what $\frac{1}{4}$ of them cost him. What per cent of the cost does he gain or lose? what per cent of the selling price?

4. Four fourths exceeds three fourths by what per cent?

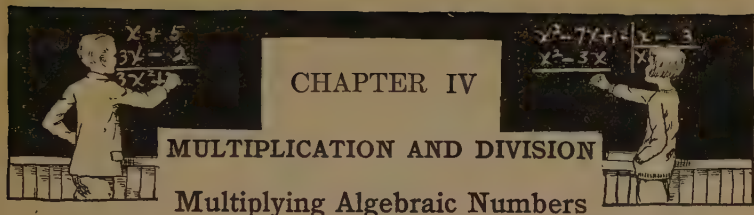
5. Agnes is 36 yr. old. She is 3 times as old as Cora was when Agnes was as old as Cora is now. How old is Cora?

6. Stanley Harrison's farm is $\frac{1}{4}$ of a square mile and Joe Lee's farm is $\frac{1}{4}$ of a mile square. The two farms are of the same quality and together sell for \$8000. How should the money be divided between the two owners?

7. Frank asked the time of day. Howard replied that 2 hr. ago the time past noon was $\frac{1}{3}$ of the time to midnight 2 hr. hence. What time was it?

8. A flagpole 44 ft. long was broken into two pieces such that if the shorter part were increased by 3 ft. and the other part were decreased by 5 ft., the first part would be $\frac{1}{2}$ as long as the second part. What was the length of each of the two pieces?

9. A farmer found that a certain triangular field required 90 rd. of fencing. The second side required 10 rd. more than the first side, and the third side required $\frac{1}{2}$ as much as the sum of the other two sides. What was the length of each side?



You will have as much occasion to multiply and divide positive and negative numbers as to add and subtract them. You must therefore understand clearly how this is done.

(1) Suppose that George is earning \$2 a day. After 3 days the number of dollars he has may be represented by:

$$3 \times 2, \text{ or } (+2) + (+2) + (+2), \text{ or } 6$$

Multiplication of a positive number, b , by a positive number, a , may be considered as a short method of finding the sum of b taken a times.

Thus, $(+a)(+b) = b + b + b + \dots$ to a times $= ab$.

(2) Suppose that George is spending \$2 a day. Since spending is the *opposite* of earning, the number of dollars he spends in 3 days may be represented by:

$$3 \times (-2), \text{ or } (-2) + (-2) + (-2), \text{ or } -6$$

Multiplication of a negative number, $-b$, by a positive number, a , is a short method of finding the sum of $-b$ taken a times.

Thus, $a \times (-b) = (-b) + (-b) + (-b) + \dots$ to a times $= -ab$.

(3) Suppose again that George has been earning \$2 a day, then his financial standing 3 days ago, compared with his standing to-day, may be represented by:

$$-3 \times (+2), \text{ or } -(+2) - (+2) - (+2), \text{ or } -6$$

That is, 3 days ago he had \$6 less than he has now.

Multiplication of a positive number, b , by a negative number, $-a$, is a short method of subtracting b a times.

Thus, $(-a)(b) = -(+b) - (+b) - (+b) \cdots$ to a times $= -ab$.

(4) Suppose again that George has been spending \$2 a day, then 3 days ago he was \$6 better off than he is now.

Thus, $(-3)(-2) = -(-2) - (-2) - (-2) = +2 + 2 + 2 = 6$, the number of dollars George had 3 days ago more than he has now.

Multiplication of a negative number, $-b$, by a negative number, $-a$, is a short method of subtracting $-b$ a times.

Thus, $(-a)(-b) = -(-b) - (-b) - (-b)$ to a times $= +b + b + b + \cdots$ to a times $= ab$.

Laws of signs in multiplication.

The product of two numbers having like signs is positive.

The product of two numbers having unlike signs is negative.

NOTE. A change of the sign of one of the factors of a product changes the sign of the product.

Consider the product of $(-2) \times (-3) \times (-4)$. The product of $(-2)(-3) = +6$. The product of $(+6)(-4) = -24$.

Similarly, $(-2)(-2)(-2)(-2) = +16$.

ORAL EXERCISES

Find the product of each of the following:

1. 5×8

3. -5×8

5. $4 \times (-2\frac{1}{2})$

2. $5 \times (-8)$

4. $-5 \times (-8)$

6. $(-3)^2$

7. $(-3)^3$

10. $(-5)^2(-3)$

8. $(-3)^4$

11. $3(-2)^2$

9. $(-2)(-2)(-3)$

12. $(-2)^3(-3)^2$

If $a = 2$, $b = -3$, $x = -2$, and $y = 3$, find the value of:

13. $a^2 + b^2$

16. $x^2 - y^2$

19. $a(x - y)$

14. $a^2 - b^2$

17. $ab - xy$

20. $b(x + y)$

15. $ax - by$

18. $ab + xy$

21. $ab(x^2 - y)$

Select from the numbers given at the right the roots of the equation at the left:

- | | |
|------------------------|----------------|
| 22. $x^2 + x - 6 = 0$ | 1, 2, 0, - 3 |
| 23. $x^2 + x = 2$ | 1, - 1, 2, - 2 |
| 24. $x^2 - x - 2 = 0$ | 0, - 1, 2, - 2 |
| 25. $x^2 + 5x + 6 = 0$ | 2, - 2, 3, - 3 |
| 26. $x^2 - 5x = 6$ | - 1, - 2, 3, 6 |

27. State the law of signs involved in the product of an even number of negative factors.

28. State the law of signs involved in the product of an odd number of negative factors.

Multiplying monomials by monomials. You have noticed that *the product of an odd number of negative factors is negative and that all other products are positive.*

Since multiplications may be performed in any order, it follows that:

$$\begin{aligned}(2a)(3b) &= 2 \times 3 \times a \times b = 6ab \\ (2a)(-3b) &= 2 \times (-3) \times a \times b = -6ab \\ (-2a)(3b) &= -2 \times 3 \times a \times b = -6ab \\ (-2a)(-3b) &= (-2) \times (-3) \times a \times b = 6ab\end{aligned}$$

The product of two monomials is found by prefixing the product of the numerical coefficients to the product of the literal factors.

The product of monomials containing the same letters. x^2 means xx , x^3 means xxx , and x^n means $xxx \cdots$ to n factors (see page 7). From this meaning of exponents, we have:

$$\begin{aligned}(x^2)(x^3) &= (xx)(xxx) = x^{2+3} = x^5 \\ (a^2x^3)(ax^2) &= (aa)(a)(xxx)(xx) = a^{2+1}x^{3+2} = a^3x^5 \\ (x^m)(x^n) &= (xxx \cdots \text{to } m \text{ factors})(xxx \cdots \text{to } n \text{ factors}) = x^{m+n}\end{aligned}$$

The exponent of a letter in a product is equal to the sum of the exponents of that letter in the factors.

Since $(x^2)^3 = (x^2)(x^2)(x^2) = (xx)(xx)(xx) = x^6$, we know that $(x^2)^3 = x^6$. Similarly, $(x^m)^n = x^{mn}$.

ORAL EXERCISES

Give the products :

- | | | |
|----------------------------------|----------------------------|---------------------|
| 1. $(2a)(3a)$ | 5. $y^3 \cdot y^8$ | 9. $(-2a)^2(3a^4)$ |
| 2. $(4a)(a^2)$ | 6. $y^4 \cdot y^4$ | 10. $5a(-7a^3)$ |
| 3. $x \cdot x^5$ | 7. $3x^2 \cdot 2x^5$ | 11. $3a \cdot 2a^3$ |
| 4. $x^2 \cdot x^4$ | 8. $(-a)(a^3)$ | 12. $2a(-3ab)$ |
| 13. $(-3a^2)(-5abc)$ | 14. $(-8a^3)(2bc)$ | |
| 15. $x \cdot x^3$ | 19. $3^2 \times 3^3$ | 23. $(2a^2)^2$ |
| 16. 2×2^3 | 20. $5^2 \times 5^4$ | 24. $(3x^3)^2$ |
| 17. $a^2 \times a^3$ | 21. 5×5^3 | 25. $(a^2)^3$ |
| 18. $2^2 \times 2^3$ | 22. 10×10^5 | 26. $(2a^3)^2$ |
| 27. $(15xy)(-4x^2y^5z)$ | 34. $7x^2y^3z(-3xy^2)$ | |
| 28. $(-3x^3)^2(-2x)$ | 35. $-(3x^3y^5)^2$ | |
| 29. $a \cdot a^3 \cdot a^2$ | 36. $(-x)^2(-x)^3$ | |
| 30. 3×3^4 | 37. $-3(-5x^2)^3$ | |
| 31. $24a \times \frac{3}{4}a^3$ | 38. $(4a^3)^2$ | |
| 32. $2a^2 \times \frac{3}{2}a^5$ | 39. $(3a^2b)^3$ | |
| 33. $3x^5(-\frac{4}{3}xy^2)$ | 40. $2(\frac{3}{2}a)(-2b)$ | |

If $a = 2$, $b = -3$, $c = -1$, find the value of :

41. $2a^3$ 42. $(2a)^3$ 43. $2a^2b^2c^2$ 44. $-abc$ 45. $(-abc)^2$

The number of literal factors in a term is called the **degree** of that term.

Thus, xy is of the *second degree*, x^3 is of the *third degree*, and $2x^2y^3$ is of the *fifth degree*.

The degree of an expression is the same as that of the term of the expression which has the highest degree.

Thus, the expression $5x^2 - 2x + 3$ is of the *second degree*.

Arrangement of terms. It is generally convenient to arrange the terms of an expression according to the ascending or the descending powers of some common letter.

Thus, the expression $x^3 - 2x^2y + 3xy^2 - y^3$ is arranged according to the descending powers of x or the ascending powers of y .

ORAL EXERCISES

Arrange the following polynomials according to the descending powers of x :

1. $5x - x^2 + 7 + x^3$

4. $3x^2 + y^2 + 2xy$

2. $3 - 7x + x^4 - x^2$

5. $4x^3 + y^3 - 3xy^2 + 2x^2y$

3. $5x^4 - 3x^2 - 6x + x^3$

6. $5x^4 - 3y + 2x^2y^2$

7. Arrange $a^2x + a^3 - ax^2 + x^3$ according to the descending powers of a .

8. State the degree of each term in ex. 1 to 3.

9. State the degree of each polynomial in ex. 4 to 6.

Multiplication of polynomials by monomials. You have seen that the product of a and $x + y + z$ is obtained by multiplying each term of the polynomial by the monomial (page 45). Such a product may be indicated as follows:

$$a(x + y + z) = ax + ay + az$$

EXAMPLE. Multiply $3xy^2 - 2x^2y + x^3 - y^3$ by $-2xy^2$.

SOLUTION. First arrange the terms according to the descending powers of x and indicate the multiplication by means of parentheses. Since the product is easily seen, write it after the indicated multiplication, thus:

$$-2xy^2(x^3 - 2x^2y + 3xy^2 - y^3) = -2x^4y^2 + 4x^3y^3 - 6x^2y^4 + 2xy^5.$$

Or, the problem may be arranged with the multiplier below the multiplicand, thus:

$$\begin{array}{r} x^3 - 2x^2y + 3xy^2 - y^3 \\ - 2xy^2 \\ \hline - 2x^4y^2 + 4x^3y^3 - 6x^2y^4 + 2xy^5 \end{array}$$

ORAL EXERCISES

Find the product of each of the following:

1. $2(a - 3b)$

8. $-5xy(x^2 - y^2)$

2. $5(x - 5y)$

9. $-5ab(2a^2 - b)$

3. $-3(a^2 - ab)$

10. $-\frac{1}{2}a(4a - 6b)$

4. $2x(x^2 - x)$

11. $\frac{1}{3}x(6x^2 - 3xy + y^2)$

5. $2a(a^2 + 2)$

12. $-p^2q(-p^2 + pq + q^2)$

6. $3xy(2x + 3y)$

13. $mn(m^2 + n^2 - 2mn)$

7. $2a^2(a^2 - b^2)$

14. $-2ab^2(3ab - b^2)$

Find the indicated products:

$$15. \ 5xy(-3x^2 - y^2 + 2xy) \quad 18. \ -a(a - 5 + 6a^2)$$

$$16. \ -7m^2n^2(mx^2 - mnx) \quad 19. \ a^3(2 + a + a^2)$$

$$17. \ x(8 - 7x + x^2) \quad 20. \ -a^2(a^2 + 2a - 1)$$

21. What is the area of a square whose side is $2s$ inches?

22. What is the area of a rectangle with a base of b inches and an altitude of $(b + 3)$ inches? an altitude of $(2b - 3)$ inches?

23. Find the area of a parallelogram if the altitude is h inches and the base is $(3h - 5)$ inches.

24. What is the area of a triangle if the base is b inches and the altitude is $(b - 5)$ inches?

25. What is the area of a trapezoid if the upper base is 30 in., the lower base is $(2b + 5)$ inches, and the altitude is $2a$ inches?

The product of two polynomials. The work of finding the product of two polynomials is much like that of finding the product of two arithmetic numbers each of which contains more than one figure; for example, 21×32 .

$$\begin{array}{r} 32 \\ 21 \\ \hline 64 \\ 672 \end{array} \qquad \begin{array}{r} 30 + 2 \\ 20 + 1 \\ \hline 30 + 2 \\ 600 + 40 \\ \hline 600 + 70 + 2 \end{array}$$

EXAMPLE 1. Find the product of $(2x + 3)$ by $(x + 5)$.

SOLUTION. In this example, each term of $2x + 3$ is to be multiplied by each term of $x + 5$. This work may be indicated thus:

$$(x + 5)(2x + 3) = x(2x + 3) + 5(2x + 3)$$

As in arithmetic, write the multiplier below the multiplicand and multiply each term of the multiplicand by each term of the multiplier. In algebra, however, it is more convenient to begin multiplying at the *left*.

$$\begin{array}{r} 2x + 3 \\ x + 5 \\ \hline x(2x + 3) = 2x^2 + 3x, \quad \text{first partial product} \\ 5(2x + 3) = 10x + 15, \quad \text{second partial product} \\ (x + 5)(2x + 3) = 2x^2 + 13x + 15, \quad \text{sum of the partial products} \end{array}$$

EXAMPLE 2. Find the product of $(3x + 2y)$ by $(2x - y)$.

SOLUTION.

$$\begin{array}{r}
 3x + 2y \\
 2x - y \\
 \hline
 2x(3x + 2y) = 6x^2 + 4xy \\
 -y(3x + 2y) = -3xy - 2y^2 \\
 \hline
 (2x - y)(3x + 2y) = 6x^2 + xy - 2y^2
 \end{array}$$

CHECK. $x = 2, y = 3$

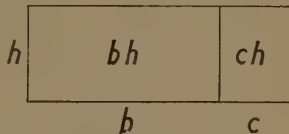
$$\begin{array}{rcl}
 = 6 + 6 & = & 12 \\
 = 4 - 3 & = & 1
 \end{array}$$

$$= 24 + 6 - 18 = 12$$

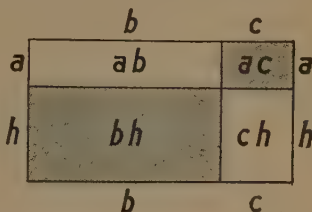
The work may be checked by substituting small values for x and y , as shown above.

The representation of products by the areas of rectangles, as shown below, may help you to understand the product of two binomials.

You have learned that the area of a rectangle is equal to the product of its base by its height or altitude, that is, $A = bh$. Suppose you wish to find the area of the rectangle whose base is $b + c$ and whose altitude is h . This area is represented by the rectangle at the right. According to the formula, the area is $h(b + c)$, or $bh + ch$, which agrees with the sum of the areas of the two smaller rectangles into which the given rectangle is divided.



Now consider a larger rectangle with the altitude $a + h$ and the base $b + c$. The area of this rectangle equals the product of its altitude and base, or $(a + h)(b + c)$. This is equivalent to the product of $h(b + c)$, as shown above, increased by the product of $a(b + c)$. The product may be written $(a + h)(b + c) = a(b + c) + h(b + c) = ab + ac + bh + ch$. This product agrees with the area of the given rectangle, which is composed of the areas of the four smaller rectangles shown in the diagram.



EXERCISES

Find the indicated products and check :

- | | |
|------------------------|----------------------------------|
| 1. $(x + 1)(x + 2)$ | 11. $(2x + 3)(x + 2)$ |
| 2. $(x - 1)(x + 2)$ | 12. $(7p + 5q)(2p - 5q)$ |
| 3. $(a + 3)(a + 2)$ | 13. $(2x^2 - y)(x^2 + y)$ |
| 4. $(a - 2)(3a + 1)$ | 14. $(7x + y^2)(3x - 2y^2)$ |
| 5. $(x - 2)(x - 3)$ | 15. $(5xy + 3)(2xy - 1)$ |
| 6. $(2x - 1)(2x + 1)$ | 16. $(4x - 3y)(5x - 2y)$ |
| 7. $(3a - 2)(3a + 2)$ | 17. $(2x - y)(\frac{1}{2}x + y)$ |
| 8. $(2a + b)(a + b)$ | 18. $(p + 2q)(2p - 5q)$ |
| 9. $(2a - 3b)(a - 2b)$ | 19. $(5r^2 - 3s)(2r^2 + 3s)$ |
| 10. $(x + y)(2x + 3y)$ | 20. $(x^2 + x)(x + 1)$ |

21. Multiply $x^2 + 1 - 2x$ by $x - 3 + x^2$. Check with $x = 2$.

SOLUTION. Arrange the terms of the multiplicand and the multiplier according to the descending powers of x .

	CHECK. $x = 2$	
$x^2 - 2x + 1$	$= 4 - 4 + 1$	$= 1$
$x^2 + x - 3$	$= 4 + 2 - 3$	$= 3$
$x^4 - 2x^3 + x^2$		
$\quad x^3 - 2x^2 + x$		
$\quad \quad - 3x^2 + 6x - 3$		
$x^4 - x^3 - 4x^2 + 7x - 3$		

$= 16 - 8 - 16 + 14 - 3 = 3$

Multiply :

- | | |
|--|----------------------------------|
| 22. $2x + x^3 - 1$ by $x + 2$ | 24. $x^2 - 3 + x$ by $x + 3$ |
| 23. $x^2 + x - 2$ by $2x - 1$ | 25. $x^2 + y^2 - 2xy$ by $x - y$ |
| 26. $3xy + x^2 - 2y^2$ by $2x + 3y$ | |
| 27. $3x^2 + 2xy - y^2$ by $3x - 5y$ | |
| 28. $12y^2 - 7xy + 2x^2$ by $-4x - y$ | |
| 29. $x^2 + y^2 - 3xy$ by $2x + y$ | |
| 30. $x^3 + x + x^2 + 1$ by $x - 1$ | |
| 31. $(x^2 - x + 1)(x^2 + x + 1)$ | |
| 32. $(x^2 - xy + y^2)(x^2 + xy + y^2)$ | |
| 33. $(2x^2 - x - 1)(x^2 + 2x - 2)$ | |

Find the indicated products :

$$34. (x^4 - x^3 + 1)(x^2 + x + 1)$$

$$35. (a^2 + 2a^3 - a + 1)(3 + a)$$

$$36. (2x - 1)(x + 2)(x - 1)$$

$$37. (a + b)(a - b)(a^2 + b^2)$$

$$38. (3x + 5)(2x - 1)(1 + 2x)$$

$$39. (2a - b)(a + 2b)(a^2 - 2ab - b^2)$$

$$40. (a + b)^2$$

$$42. (2a - 3b)^2$$

$$44. (a - b)^3$$

$$41. (2a - b)^2$$

$$43. (3a - b)^2$$

$$45. (a + b)^3$$

Equations Applying Multiplication

It would not be worth while to learn the principles of algebra if they did not have useful applications. Multiplication of the kind that you have just studied is often necessary for the solution of equations, and equations, as you know, are useful tools in solving many kinds of problems.

Suppose, for example, that you wish to solve this problem :

The height of a certain rectangle is 3 in. longer than its base. If the area of this rectangle were 1 sq. in. smaller, it would equal the area of a square having a side 1 in. longer than the base of the rectangle. Find the dimensions of each figure.

SOLUTION. Let b = the number of inches in the base of the rectangle

Then $b + 3$ = the number of inches in the height of the rectangle

and $b + 1$ = the number of inches in the side of the square.

The equation of the problem is: $b(b + 3) - 1 = (b + 1)^2$ (1)

Perform the indicated multiplications: $b^2 + 3b - 1 = b^2 + 2b + 1$ (2)

Subtract $b^2 + 2b$ from both sides of (2): $b - 1 = 1$ (3)

$$\text{or } b = 2$$

$$\text{Hence } b + 3 = 5$$

$$\text{and } b + 1 = 3$$

The rectangle is 2 in. by 5 in. and the square is 3 in.

CHECK. The area of the rectangle is 2×5 sq. in., or 10 sq. in.

The area of the square is 3×3 sq. in., or 9 sq. in., which is 1 sq. in. less than the area of the rectangle.

EXERCISES

Solve the following equations. Check ex. 1 to 10:

1. $3(2x - 7) = 2x + 7$
2. $5(x - 3) = 3x + 1$
3. $7(2x + 5) = 2x + 41$
4. $x(x - 2) = x^2 - 6$
5. $x(2x + 3) - 3 = 2x^2$
6. $(2x - 1)(x - 5) = 2(3 + x^2)$
7. $(x - 5)(2x + 1) = 2x(x - 4)$
8. $x^2 - 3(x - 2) = x(x + 3)$
9. $x^2 - 5x - x(x - 7) = -4$
10. $(x - 2)(x + 1) = x^2 + 3$
11. $(x + 8)(x - 4) = 7 - (x + 5)(3 - x)$
12. $3(x^2 - 1) - 3(x - 1)^2 = 15 - x$
13. $(2n + 3)(n + 3) - (n + 1)(2n + 1) = 14$
14. $(x - 2)(x - 5) = (x + 1)(x + 3) - 4$
15. $(2x - 3)(x - 7) + 15 = (x - 1)(x + 4) + x^2$
16. $(x + 3)^2 - 2(5x + 6) = (x - 1)^2$
17. $(x + 2)^2 - (2x + 1)^2 = 3x(1 - x)$
18. $2x(4x + 3) + 4 = (3x - 2)^2 - (3 - x)^2$
19. $(x + 1)^2 + (x + 2)^2 - (x - 3)^2 = x^2$
20. $(2x + 1)^2 + (2x - 1)^2 - (3x - 2)^2 = 8 - x^2$
21. $(n + 2)^2 - (n + 3)^2 = -17$
22. $n(4n + 5) - (2n - 1)^2 = 11$
23. $(x + 3)^2 - (2x - 5)^2 = 3x(8 - x)$
24. $(x + 3)^2 - x(x + 5) = 15$
25. $(2x - 3)^2 - (1 + 4x^2) = 12$
26. $(x - 5)(x + 3) - x(x - 7) = 5$
27. $(2x - 3)(x + 2) - 2(x - 3)^2 = 2$
28. $2x(x - 7) = 22 + (x + 3)(2x - 1)$
29. $(7x - 5)(x + 2) - (3x + 2)(2x + 1) = x^2 + 4$
30. $(3x + 5)(2x - 7) - (4x + 3)(2x - 1) = x(3 - 2x)$

Dividing Algebraic Numbers

In algebra, as in arithmetic, division is the opposite or inverse of multiplication. In multiplication, two numbers are given and a third number, called their product, is sought. This may be represented by $a \times b = p$. In division, the product and one of the numbers are given and the other number is required. This may be represented by $D \div d = q$, in which D is the dividend, d , the divisor, and q , the quotient.

Division is commonly expressed with the division sign, as above, or in the form of a fraction: $\frac{D}{d} = q$. (See page 6.)

The following similarities in regard to signs should be noted:

$$(+a) \times (+b) = +p; \text{ hence } +p \div (+b) = +a$$

$$(+a) \times (-b) = -p; \text{ hence } -p \div (-b) = +a$$

$$(-a) \times (+b) = -p; \text{ hence } -p \div (+b) = -a$$

$$(-a) \times (-b) = +p; \text{ hence } +p \div (-b) = -a$$

Law of signs in division.

If the dividend and the divisor have like signs, their quotient is positive; but if they have unlike signs, their quotient is negative.

ORAL EXERCISES

Perform the indicated division:

- | | | |
|--------------------|---------------------|--------------------------|
| 1. $18 \div (-3)$ | 4. $-155 \div (-5)$ | 7. $-18 \div -36$ |
| 2. $-18 \div (-3)$ | 5. $-56 \div 8$ | 8. $-9 \div 27$ |
| 3. $-27 \div 9$ | 6. $77 \div -7$ | 9. $-\frac{3}{4} \div 3$ |

Dividing monomials by monomials.

$$a^3 \times a^2 = (a \cdot a \cdot a)(a \cdot a) = a^{3+2} = a^5. \quad (\text{See page 59.})$$

$$\text{Hence } a^5 \div a^2 = \frac{\overset{11}{\cancel{aaaaa}}}{\underset{11}{\cancel{aa}}} = a^{5-2} = a^3$$

$$\text{Similarly, } \frac{x^m}{x^n} = \frac{\overset{11}{\cancel{xxxx \dots \text{to } m \text{ factors}}}}{\underset{11}{\cancel{xxx \dots \text{to } n \text{ factors}}}} = x^{m-n}$$

The quotient of two monomials is obtained in a way similar to that of arithmetic :

$$\frac{36}{12} = \frac{\overset{1}{2} \times \overset{1}{2} \times \overset{1}{3} \times 3}{\underset{1}{2} \times \underset{1}{2} \times \underset{1}{3}} = 3$$

NOTE. When it is said that like factors are *canceled* from both the dividend and the divisor, they are not *canceled* in the sense that the result is zero, but they are divided into each other with the quotient 1, thus :

$$\frac{\overset{1}{3}}{\underset{1}{3}} = 1, \frac{\overset{1}{3} \times \overset{1}{3}}{\underset{1}{3} \times \underset{1}{3}} = 1; \text{ also } \frac{a^2}{a^2} = \frac{\overset{1}{a} \times \overset{1}{a}}{\underset{1}{a} \times \underset{1}{a}} = 1, \text{ or } \frac{a^2}{a^2} = a^{2-2} = a^0 = 1$$

Any number with an exponent 0 is equal to 1.

Instead of actually canceling factors in division, the work may be expressed thus :

$$\frac{a^5}{a^2} = a^{5-2} = a^3 \quad \frac{2^5}{2^2} = 2^{5-2} = 2^3 \quad \frac{-15 a^4 b^3}{5 a^2 b} = -3 a^2 b^2$$

The exponent of any letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

The sign of the quotient is + or -, according to whether the dividend and the divisor have like signs or unlike signs.

EXERCISES (1 to 12 Oral)

Divide :

1. $\frac{5^4}{5^2}$

5. $\frac{-a^3}{a^2}$

9. $\frac{-15 a^3 x}{3 a x}$

2. $\frac{4^3}{4}$

6. $\frac{x^3}{-x}$

10. $\frac{-14 x^3 y^2}{-7 x y^2}$

3. $\frac{2^3}{2}$

7. $\frac{-x^6}{-x^5}$

11. $\frac{-21 a x^2 y^3}{7 a x^2 y^3}$

4. $\frac{-3^4}{3}$

8. $\frac{a^3 b^2}{ab}$

12. $\frac{-27 x^3 y^2 z}{-9 x y^2 z}$

Find the indicated quotients:

$$13. \quad \frac{-12 a^2 b}{18 a b^2} \quad \text{SOLUTION.} \quad \frac{-12 a^2 b}{18 a b^2} = -\frac{\overset{1}{\cancel{6}} \times 2 \overset{1}{\cancel{a}} \overset{1}{\cancel{b}}}{\overset{1}{\cancel{6}} \times 3 \overset{1}{\cancel{a}} \overset{1}{\cancel{b}} b} = -\frac{2 a}{3 b}$$

$$14. \quad \frac{8 a b^3}{12 a} \qquad 17. \quad \frac{15 a x y^2}{-25 a x^2 y} \qquad 20. \quad \frac{39 a^2 b^3}{-52 a b}$$

$$15. \quad \frac{9 x^4 y}{18 x^2 y^2} \qquad 18. \quad \frac{-64 r s^5}{16 s^6} \qquad 21. \quad \frac{-42 a x^3}{28 x}$$

$$16. \quad \frac{25 m n^4}{-5 n^5} \qquad 19. \quad \frac{12 x^4 y^7}{-6 x^7 y^7} \qquad 22. \quad \frac{-8 a b^4}{18 a^2 b}$$

Dividing polynomials by monomials. In finding the product of a and $x + y$, each term of $x + y$ is multiplied by a . (See page 61.) Hence $a(x + y) = ax + ay$.

The quotient of $ax + ay$ divided by a is obtained by the reverse process, that is, by dividing each term of $ax + ay$ by a .

$$\frac{ax + ay}{a} = \frac{ax}{a} + \frac{ay}{a} = x + y$$

Similarly,

$$\frac{a^4 x^2 + a^3 x^3 + a^2 x^4}{a^2 x^2} = \frac{a^4 x^2}{a^2 x^2} + \frac{a^3 x^3}{a^2 x^2} + \frac{a^2 x^4}{a^2 x^2} = a^2 + ax + x^2$$

The quotient of a polynomial divided by a monomial is equal to the sum of the quotients of each term of the polynomial divided by the monomial.

EXERCISES (1 to 10 Oral)

Find the indicated quotients:

$$1. \quad (2r + 6) \div 2$$

$$2. \quad (3r^2 - qr) \div 3r$$

$$3. \quad (15a - 12) \div -3$$

$$4. \quad (16bx - 12by) \div 4b$$

$$5. \quad (12a^2 - 8ab) \div -4a$$

$$6. \quad (21ab - 7b^2) \div 7b$$

$$7. \quad (25a^3 - 15a^2) \div 5a$$

$$8. \quad \frac{15x^2 - 6x}{3x}$$

$$9. \quad \frac{15x^2 - 3x}{3x}$$

$$10. \quad \frac{6a^2 - 2a}{2a}$$

Find the indicated quotients :

$$11. \frac{28 a^4 - 7 a^2}{7 a}$$

$$12. \frac{45 a^2 x^3 - 27 a x}{-9 a x}$$

$$13. \frac{2 a^3 x^3 - 3 a^2 x^2 + 4 a x}{-a x}$$

$$14. \frac{15 n^5 - 20 n^3 + 10 n^2}{5 n^2}$$

$$15. \frac{18 x^4 - 30 x^3 + 12 x^2 + 6 x}{6 x}$$

$$16. \frac{18 a^4 + 54 a^2 + 36 a}{18 a}$$

$$17. \frac{75 m^3 - 50 m^4 + 25 m^5}{25 m^2}$$

$$18. \frac{72 a^2 b - 36 a b + 18 a b^2}{18 a b}$$

$$19. \frac{a^4 x^4 - a^2 x^2 + a x}{-a x}$$

$$20. \frac{14 x^3 - 21 x^4 + 28 x^2}{7 x}$$

$$21. \frac{27 x y z - 36 x^2 y^2 z^2 + 45 x^3 y^3 z^3}{9 x y z}$$

$$22. \frac{20 x^4 y^2 - 16 x^2 y^3 + 12 x y^4}{4 x y}$$

$$23. \frac{5(x + y) - a(x + y)}{(x + y)}$$

$$\text{SOLUTION. } \frac{\frac{1}{5(x+y)} - \frac{1}{a(x+y)}}{\frac{1}{(x+y)}} = 5 - a$$

$$24. \frac{ax(2a + b) - by(2a + b)}{2a + b}$$

$$25. \frac{m(x - y) - 2n(x - y)}{x - y}$$

$$26. \frac{a^2(x^2 - y^2) - b^2(x^2 - y^2)}{x^2 - y^2}$$

$$27. \frac{(a + x)^3 - 3(a + x)^2 + 5(a + x)}{a + x}$$

$$28. \frac{-3(a - 2x)^3 + 5(a - 2x)^2 - (a - 2x)}{-(a - 2x)}$$

$$29. \frac{5ab(c - d) - 30a^2(c - d) + 15ac(c - d)}{5a(c - d)}$$

Study the following example :

SOLUTION.

$$\begin{array}{r} 4 \times 23 = \quad 92 \\ \text{Subtract:} \quad \underline{161} \\ 7 \times 23 = \quad 161 \end{array}$$

EXAMPLE 2. Divide $x^2 + 6x + 8$ by $x + 2$. Check.

SOLUTION. $x^2 + 6x + 8$ $\overline{) \begin{array}{l} x + 2 \\ x + 4 \end{array}}$ $x(x + 2) = \underline{x^2 + 2x}$ Subtract: $\begin{array}{r} 4x + 8 \\ 4x + 8 \end{array}$ $4(x + 2) = \underline{\hspace{1cm}}$	CHECK. $x + 4$ $\underline{x + 2}$ $x^2 + 4x$ $\quad + 2x + 8$ $\underline{\hspace{1cm}}$ $x^2 + 6x + 8$
--	--

$$(x+4)(x+2) = x(x+2) + 4(x+2) = x^2 + 2x + 4x + 8 = x^2 + 6x + 8$$

In finding the quotient of $x^2 + 6x + 8$ by $x + 2$, the first number is obtained by dividing the first term of the dividend by the first term of the divisor. Thus, $x^2 \div x = x$.

The first partial product is $x(x + 2)$, or $x^2 + 2x$. This is subtracted from the dividend, and the remainder, $4x + 8$, becomes a new dividend.

The process is repeated. x of the divisor is contained in $4x$ of the dividend 4 times. 4 is written as the second term of the quotient. The second partial product is $4(x + 2)$, which is subtracted from the dividend. The remainder is 0. The check by multiplication at the right of the work shows that it is correct.

The natural method of checking an exercise in division is to multiply the divisor by the quotient. Since $\frac{D}{d} = q$ (page 67), then $D = dq$. The work may be checked also by substitution. If this is done, care must be taken to select values for the letters that will not make the divisor zero, as division by zero is impossible.

$$\text{Thus, if } x = 2, \quad \frac{x^2 + 6x + 8}{x + 2} = \frac{4 + 12 + 8}{2 + 2} = \frac{24}{4} = 6.$$

Also if $x = 2$, the quotient $x + 4 = 2 + 4 = 6$.

The divisor may be placed at the left of the dividend or at the right, as on page 71. It is placed at the right of the dividend for the sake of convenience in finding the partial products.

NOTE. To divide one polynomial by another polynomial, arrange the dividend and the divisor in either the descending or the ascending powers of some common letter.

If there is a remainder, write it over the divisor in the form of a fraction and add the fraction to the quotient.

EXAMPLE 3. Divide $x^2 + 6x^2 - 2$ by $2x - 1$.

SOLUTION. First arrange the dividend according to the descending powers of x .

	CHECK. $x = 2$	
$6x^2 + x - 2$	$2x - 1$	$6x^2 + x - 2 = 24$
$3x(2x - 1) = 6x^2 - 3x$	$3x + 2$	$2x - 1 = 3$
Subtract:	$4x - 2$	$24 \div 3 = 8$
$2(2x - 1) = 4x - 2$	$4x - 2$	$3x + 2 = 8$

EXERCISES

Divide and check:

- | | |
|--|---|
| <p>1. $x^2 + 5x + 6$ by $x + 3$</p> <p>2. $x^2 + 7x - 8$ by $x - 1$</p> <p>3. $x^2 + x - 12$ by $x + 4$</p> <p>4. $x^2 + 2x - 15$ by $x + 5$</p> <p>5. $x^2 - 8x - 9$ by $x + 1$</p> <p>6. $x^2 + 5x + 4$ by $x + 4$</p> <p>7. $x^2 - 6x + 8$ by $x - 2$</p> <p>8. $y^2 - 9y + 8$ by $y - 1$</p> | <p>9. $6x^2 - 5x - 6$ by $3x + 2$</p> <p>10. $10x^2 - 29x - 21$ by $2x - 7$</p> <p>11. $2x^2 + 9x - 35$ by $2x - 5$</p> <p>12. $10x^2 - 6 + 11x$ by $2x + 3$</p> <p>13. $11x - 14 + 15x^2$ by $3x - 2$</p> <p>14. $6x^2 + 15x - 21$ by $7 + 2x$</p> <p>15. $6x^2 - 11x - 7$ by $2x + 1$</p> <p>16. $ax - xy + by - ab$ by $x - b$</p> <p>17. $10ax + 4bx + 6by + 15ay$ by $5a + 2b$</p> |
|--|---|

36. $x^3 - 49x - 110$ by $x + 5$ **41.** $32x^5 - 1$ by $2x - 1$

Equations with Fractional Coefficients

Very often the conditions of a problem give rise to equations having fractional coefficients, as in the case of the problem, "If $\frac{1}{3}$ of my money is \$4, how much money have I?" Let us see how such equations are solved.

Evidently in the equation $\frac{x}{3} = 4$ the first thing to do is to get rid of the denominator 3. This is done by multiplying both sides of the equation by 3, thus:

$$3\left(\frac{x}{3}\right) = 3 \times 4, \text{ or } x = 12 \quad (\text{See page 18.})$$

Similarly, the equation $\frac{x}{2} - \frac{x}{3} = 3$ may be solved by multiplying both sides by some number that will get rid of the denominators. The 2 may be removed by multiplying both sides by 2, and the 3 by multiplying both sides by 3. The double multiplication is equivalent to a single multiplication of both sides of the equation by 6, thus:

$$6\left(\frac{x}{2}\right) - 6\left(\frac{x}{3}\right) = 6 \times 3, \text{ or } 3x - 2x = 18. \quad \text{That is, } x = 18.$$

EXAMPLE. Solve the equation $\frac{2}{3}(x - 3) + 5 = \frac{3}{4}(x - 1) + \frac{x}{3}$.

Check the answer.

SOLUTION.

$$\frac{2}{3}(x - 3) + 5 = \frac{3}{4}(x - 1) + \frac{x}{3} \quad (1)$$

12 is the smallest number that will contain 3 and 4 as factors. It is called the *lowest common denominator* (L. C. D.).

Multiply both sides of (1) by 12:

$$8(x - 3) + 60 = 9(x - 1) + 4x \quad (2)$$

$$\text{Clear of parentheses and combine: } 8x + 36 = 9x - 9 \quad (3)$$

$$\text{Solve (3): } x = 9$$

CHECK.

$$\frac{2}{3}(9 - 3) + 5 = \frac{3}{4}(9 - 1) + \frac{9}{3}$$

$$4 + 5 = 6 + 3$$

$$9 = 9$$

Multiply each side of the equation by the smallest number that will contain each of the denominators as a factor.

EXERCISES

Solve and check the equations:

1. $3x = 14 + \frac{1}{3}x$ 3. $\frac{1}{3}x + \frac{1}{4}x = 7$ 5. $\frac{2}{3}n - 3 = \frac{1}{2}n$

2. $\frac{1}{3}x - 2 = \frac{1}{4}x$ 4. $\frac{2}{5}x - \frac{1}{3}x = 3$ 6. $\frac{1}{2}x = 1\frac{1}{8}$

7. $\frac{x}{2} = \frac{x}{3} + 2\frac{1}{2}$

11. $\frac{2x}{3} - \frac{x}{10} = \frac{x}{5} + 5\frac{1}{2}$

8. $\frac{2}{3}n = \frac{1}{4}n + \frac{5}{3}$

12. $\frac{n}{2} - \frac{n}{3} + \frac{n}{4} = 10$

9. $\frac{3x}{2} - \frac{x}{3} = \frac{x}{4} + 11$

13. $\frac{x}{2} + \frac{2x}{3} - \frac{3x}{4} = 15$

10. $\frac{x}{4} + \frac{x}{3} = \frac{4x}{3} - \frac{3}{2}$

14. $\frac{2}{3}y + \frac{3}{4}y = \frac{3}{2}y - 2$

15. Find a number whose half and third together equal 15.

16. Find two numbers whose sum is 25 if one of them exceeds $\frac{1}{3}$ of the other by 1.17. Find two numbers whose difference is 8 if $\frac{1}{4}$ of one of them is equal to $\frac{1}{3}$ of the other.18. Ann spent $\frac{1}{3}$ of her money for ribbon and had 28¢ left. How much did she have at first?19. Henry and Lee had equal sums of money. Henry spent $\frac{1}{4}$ of his money for a radio tube. Lee spent $\frac{1}{3}$ of his for books. Together they had \$17 left. How much did each have at first?20. The population of two neighboring towns is 4300. If the population of the smaller town is 200 less than $\frac{1}{2}$ of that of the larger town, find the number of people in each town.21. Mr. Smith bought 16 tons of coal. Part of it was put into his own cellar and the rest into his son's cellar. How much did each receive if the son had $2\frac{1}{2}$ tons more than $\frac{1}{2}$ as much as the father?22. Ten years ago the price per ton of nut coal was \$1.75 less than $\frac{1}{2}$ of the price now. Find the price of coal 10 yr. ago if the cost of 8 tons now is \$16 less than 3 times the cost 10 yr. ago.

REVIEW OF CHAPTERS I TO IV

1. Find the value of $3 \times 4 - 8 \div 2 + 15 \times 6 \div 18 - 10$.
2. Find the value of $8 \times 3^2 + 7 \times 3^2 - 10 \times 3^2 + 5 \times 3^2$.

SOLUTION. Combine the terms and get $10 \times 3^2 = 90$. Explain.

3. Find the value of $17 \times 13 + 3 \times 13 - 12 \times 13 + 2 \times 13$.

4. $15 \times 23 - 18 \times 23 + 12 \times 23 + 23 = ?$

5. $5 \times 2^2 - 4 + 12 \div 3 + 5 - 9 \div 3 = ?$

6. $5(2^2 - 4) + 12 \div (3 + 5) - 9 \div 3 = ?$

7. State the order of operations to be observed in ex. 5 and ex. 6.

8. State and illustrate a way in which algebraic methods are shorter than those of arithmetic.

9. From the sum of $3x^2 + 2x - 5$ and $x^2 - 3x + 6$ subtract $5 - (4x - 2x^2)$.

10. What is the perimeter of a triangle whose sides are $2n - 3$, $n + 5$, and $3n - 7$? Find each side if the perimeter is 25 in.

11. Find the perimeter of an isosceles triangle if the base is $(2b + 3)$ inches and one of the equal sides is $(3b - 1)$ inches. Find each side if the perimeter is 49 in.

12. The dimensions of a rectangular field are $(5r + 3)$ rods and $(7r - 3)$ rods. What is the perimeter? Find each side if the perimeter is 266 rd.

13. If the three angles of a triangle are $2a^\circ$, $(3a - 5)^\circ$, and $(a + 11)^\circ$, respectively, what is the value of each angle?

14. What are roots of an equation? What is meant by satisfying an equation? Illustrate both terms by solving $3(x - 2) = 9$.

15. Give the law of signs in subtraction.

16. Illustrate two ways of checking an exercise in subtraction.

17. What must be added to $x + a$ to make $y - b$?

18. What must be subtracted from $3s + d$ to give $s - 5d$?

19. A man earns d dollars a week and spends s dollars. How much will he save in w weeks? in a year? in y years?

20. Harold bought m dozen tennis balls at d dollars a dozen and sold them at c cents each. What was the gain?

21. I have m dollars. How much interest do I receive annually on $\frac{1}{5}$ of it at 6%? on the remainder at 4%?

22. What is the interest on p dollars at 5% for y years? What is the amount?

23. State the rule for removing parentheses.

24. Simplify $3x - 5x - 3(2x + y) - 2y$.

25. Simplify $2a - 3(3b - a) + 2 - 5a - (b + 3a)$.

26. Group the x and y terms in a parenthesis:

(a) $9a^2 - x^2 - 6xy - 9y^2$ (b) $25m^2 - 49x^2 + 7xy - 25y^2$

27. Express $5x - 7ax + 3bx$ as a single term.

28. Express the x quantities as one term and the y quantities as a second term in $ax + 5y - 5x - ay$.

29. What is the law of signs in multiplication?

30. Give the value of each of the following:

(a) $(-1)^3$ (b) $(-1)^6$ (c) $2(-2)^3$ (d) $-5(-5)^2$

31. Simplify $(2a - 3b)^2 - 5a(2a - b) + 2b(3b - a)$.

32. What is the law of exponents in multiplication?

33. Simplify:

(a) $(2x)^3$ (b) $2(x)^3$ (c) $(x^5)(x)^2(x)$ (d) 2×2^3

34. Multiply $2x^2 - 3x + 6$ by $x + 3$. Check.

35. What is meant by a power of a number? by the degree of a term? by the degree of an expression?

36. In what order should the terms of an expression be arranged? Illustrate in the following expressions:

(a) $a^3 - x^3 + ax^2 - 2a^2x$ (b) $5x - 7 + 3x^3 - 2x^2$

37. A rectangle has a width of w and a length of $w + 5$. What is its area? Draw a figure to show this.

38. What is the area of a rectangle whose altitude is $a + 3$ and whose base is $a + 6$? Draw a figure to show this.

39. The perimeter of triangle ABC is 50 in. If side AB equals side AC and their sum exceeds side BC by 10 in., how long is each side?

40. Find three consecutive even integers if the first increased by twice the third equals 12 more than twice the second.

41. Find two consecutive integers such that $\frac{1}{2}$ of the smaller equals $\frac{1}{3}$ of the larger increased by 2.

42. What is the law of signs in division?

43. Find the indicated quotients:

$$(a) \frac{-27a}{3a} \quad (b) \frac{-15a}{-5a} \quad (c) \frac{12x}{6} \quad (d) \frac{9ax}{-3a}$$

44. State the law of exponents in division.

45. Find the indicated quotients:

$$(a) \frac{x^3}{x} \quad (b) \frac{(a-x)^7}{(a-x)^5} \quad (c) \frac{a^3}{a^3} \quad (d) \frac{(-3)^4}{-3}$$

46. If D denotes the dividend, d the divisor, q the quotient, and R the remainder, express D in terms of the other letters. Express q in terms of the other letters.

47. Give two methods of checking a division problem.

48. Divide $6x^4 - 3y^4 + x^2y^2 + 13xy^3$ by $3y^2 - 2xy + 2x^2$. Check.

49. Divide $9x^4 - x^2 + 18x - 15$ by $3x^2 - 2x + 5$. Check.

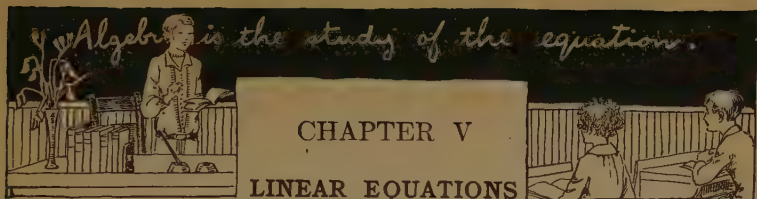
50. What is meant by cancellation in division? What is the result if all the factors in the divisor cancel with all the factors in the dividend?

51. Find each of the following indicated quotients:

$$(a) \frac{12a^2b^3}{3ab^3} \quad (b) \frac{2ax}{2ax} \quad (c) \frac{18r^3s^2}{12rs^2} \quad (d) \frac{6 \times 2^5}{3 \times 2^2}$$

52. The sum of two numbers is 37 and their difference is 9. What are the numbers?

53. Ernest and Arthur had \$7.75 between them. Ernest spent \$1.25 and Arthur spent \$2. If they then had equal amounts of money, how much did each have at first?



CHAPTER V

LINEAR EQUATIONS

Many simple problems in the preceding chapters have been solved by equations; but you must learn more about equations before you can solve more difficult problems.

The **degree** of an equation is the same as that of the term of the equation that has the highest degree.

Thus, $2x + 3 = 7$ and $5x = y$ are of the first degree, and $xy = x - 6$ is of the second degree.

An equation which in its simplest form is of the first degree is called a **linear equation**.

Identities. What is the value of x in the equation $5x - 2x = 6$? Would this equation be true for any other value of x than 2? Compare this with the value of x in $5x - 2x = 3x$. In this equation, substitute for x the value 1, 5, 10, or any other value. Thus, if $x = 1$, each side equals 3; if $x = 5$, each side equals 15, etc. In every case the two sides of the equation are *identical*; hence the equation is true for all values of x .

An equation that is true for every value of the letters involved is called an **identity**. Every equation expressed entirely in arithmetic numbers is an identity.

Since the equation $5x = 15$ is true only if the condition that $x = 3$ is fulfilled, it is not an identity, but it is a *conditional equation*. What does $5x$ equal if $x = 2$? if $x = 4$? if $x = 5$?

Show that $x^2 + 3x = 4$ is true if $x = 1$ or if $x = -4$, but that the equation is not true for other values of x .

When a root of an equation is substituted for the unknown number, the equation is *satisfied* and becomes an identity.

Thus, if you substitute 3 for x in $5x = 15$, you have $5 \times 3 = 15$, or $15 = 15$, which is an identity.

ORAL EXERCISES

Select the identities and the equations in ex. 1 to 6:

- | | |
|-------------------------------|-------------------------------|
| 1. $3x = 18$ | 4. $4x + 5 = 3x - 8$ |
| 2. $2x = x + 4$ | 5. $3x + 2x - 3 = 6x - 3 - x$ |
| 3. $5(2x - 3) = 6x - 15 + 4x$ | 6. $3(x - 1) = 8 + 3x - 11$ |

Which of the numbers at the right of the following equations are roots of the equations?

- | | |
|-----------------------------|---------|
| 7. $5x - 12 = 3$ | 2, 3, 4 |
| 8. $2n + 3 = n + 12$ | 0, 5, 9 |
| 9. $3x - 5 = x + 5$ | 3, 4, 5 |
| 10. $\frac{16}{n} = 2n - 4$ | 2, 4, 8 |
| 11. $x^2 - 5x + 6 = 0$ | 2, 3, 4 |
| 12. $n^2 = 5n$ | 0, 3, 5 |

Solve the following equations:

- | | | |
|-----------------------|-------------------------|----------------------------|
| 13. $x + 5 = 9$ | 19. $2x = x + 3$ | |
| 14. $n - 3 = 5$ | 20. $3n = 2n + 8$ | |
| 15. $4x = 24$ | 21. $5n = 3n + 4$ | |
| 16. $x + 3 = 0$ | 22. $3x = x + 4$ | |
| 17. $3x = 15$ | 23. $2(x + 3) = 8$ | |
| 18. $x - 4 = 0$ | 24. $3(n - 1) = 2n + 5$ | |
| 25. $\frac{x}{2} = 5$ | 27. $\frac{x}{4} = 3$ | 29. $\frac{3n}{4} = 9$ |
| 26. $\frac{x}{3} = 4$ | 28. $\frac{2n}{3} = 4$ | 30. $\frac{5n}{3} = n + 2$ |

31. The value of x in $\frac{x}{3} = 5$ is found by ____.
32. The value of x in $7x = 35$ is found by ____.
33. A root of an equation is ____.
34. An equation may be satisfied by ____.
35. State the axioms used in solving ex. 13; ex. 14; ex. 15; ex. 25.

Transposition. The equation $3(n - 1) = 2n + 5$ (ex. 24, page 80) was solved by means of the fundamental axioms underlying the operations performed. Notice the explanation of this work.

SOLUTION. $3(n - 1) = 2n + 5$ (1)

Perform the indicated multiplication: $3n - 3 = 2n + 5$ (2)

Add 3 and subtract $2n$ from each side: $3n - 2n = 5 + 3$ (3)

Combine like terms in (3): $n = 8$ (4)

In the second step $+ 2n$ is subtracted from both sides of equation (2) and appears as $- 2n$ on the left side of equation (3), while 3 is added to both sides of equation (2) and becomes $+ 3$ on the right side of equation (3).

Instead of saying that $2n$ in the above solution is subtracted from both sides of the equation and that 3 is added to both sides, it is sometimes said that such numbers are **transposed** to the other side of the equation with their signs changed. *Transposition*, however, is *not* an algebraic operation. It is simply a short term for expressing the operations of addition and subtraction as used in removing terms from one side of an equation to the other side. The term *transposition* is never applied to operations involving multiplication or division.

A term may be transposed from one side of the equation to the other side, provided the sign of the term is changed.

EXERCISES

1. Solve the equation $.3x + 5 = .5(34 - x)$ and check the roots.

SOLUTION. $.3x + 5 = .5(34 - x)$

Perform the indicated multiplication: $.3x + 5 = 17 - .5x$

Transpose $-.5x$ and 5 with signs changed: $.3x + .5x = 17 - 5$

Combine like terms: $.8x = 12$

Divide both sides by .8: $x = 15$

CHECK. $.3(15) + 5 = .5(34 - 15)$

$4.5 + 5 = .5 \times 19$

$9.5 = 9.5$

Solve the equations and check the roots:

2. $5(2x - 7) = 5$
3. $5(n - 3) = 15$
4. $6x + 9.6 = 3(17 - 4x)$
5. $4(2x - 3) = 3 + 13x$
6. $8 - (3x - 2) = 30$
7. $3(2x - 5) = 4x + 11$
8. $3x - 12 = 5(8 - 2x)$
9. $5(x - 2) = 3(2 - x)$
10. $3(6n + 4) = 8n + 17$
11. $2(x - 5) = 3x - 19$
12. $.5x - 2 = .3x + 2$
13. $1.7x + 3 = 13 - .8x$
14. $3(2x - 6) = 2(5x + 15)$
15. $21x = (5 + x)^2 - (x - 4)^2$
16. $5.7x - 17.2 = 9.3x - 8.2$
17. $x(x - 1) + 6 = x(x + 5) - 12$
18. $x^2 + 11 = (x + 2)(x + 1) + 3$
19. $(2x - 3)(x - 7) = (x - 1)(x + 4) + x^2 + 1$

$$20. \frac{2}{5}x - \frac{x - 6}{3} = 3$$

SOLUTION.

$$\frac{2}{5}x - \frac{x - 6}{3} = 3 \quad (1)$$

Multiply both sides by the L. C. D., which is 5×3 , or 15:

$$15 \left(\frac{2}{5}x - \frac{x - 6}{3} \right) = 15 \times 3 \quad (2)$$

$$3(2x) - 5(x - 6) = 15 \times 3 \quad (3)$$

Perform the indicated multiplication: $6x - 5x + 30 = 45$ (4)

Solve (4):

$$x = 15$$

CHECK. $\frac{2}{5}(15) - \frac{15 - 6}{3} = 3$; $6 - 3 = 3$, or $3 = 3$.

Note that parentheses are inserted around the numerator in equation (2). This is necessary to avoid mistakes in signs.

$$21. \frac{x - 1}{2} - \frac{x - 3}{2} = \frac{x - 3}{4} \quad 22. \frac{x - 1}{4} + \frac{x - 2}{3} = \frac{2}{3}(2x - 7)$$

$$23. \frac{2x - 1}{3} - \frac{2x + 1}{10} = \frac{4 - x}{5} + \frac{1}{10}$$

$$24. \frac{3x + 1}{3} + 1\frac{1}{8} = \frac{2x + 1}{4}$$

Literal Equations

In the formula $A = bh$, you can readily find the value of A when you have the value of b and of h . But in case you have the value of A and of b and wish to find the value of h , you must first change the formula to $h = \frac{A}{b}$; or if you know A and h and wish to find b , you must change the formula to $b = \frac{A}{h}$.

An equation in which some of or all the known numbers are represented by letters is called a **literal equation**.

In a literal equation like $ax = b$, which contains the first and last letters of the alphabet, the first letters usually represent known quantities and the last letters, unknown quantities.

Literal equations are solved by the same methods as numerical equations.

EXAMPLE 1. (a) Solve $3x = 6$. (b) Solve $ax = b$.

SOLUTION.

$$3x = 6$$

Divide both sides by 3: $x = \frac{6}{3}$, or 2

SOLUTION.

$$ax = b$$

Divide both sides by a : $x = \frac{b}{a}$

EXAMPLE 2. (a) Solve $\frac{x}{5} = 2$. (b) Solve $\frac{x}{c} = d$.

SOLUTION.

$$\frac{x}{5} = 2$$

Multiply both sides by 5:

$$5\left(\frac{x}{5}\right) = 5(2)$$

or

$$x = 10$$

SOLUTION. $\frac{x}{c} = d$

Multiply both sides by c :

$$c\left(\frac{x}{c}\right) = cd$$

or

$$x = cd$$

EXAMPLE 3. (a) Solve

$$3x + 7 = 19.$$

SOLUTION. $3x + 7 = 19$

Subtract 7 from each side:

$$3x = 19 - 7, \text{ or } 12$$

Divide each side by 3:

$$x = \frac{12}{3}, \text{ or } 4$$

(b) Solve $ax + b = c$.

SOLUTION.

$$ax + b = c$$

Subtract b from each side:

$$ax = c - b$$

Divide each side by a : $x = \frac{c - b}{a}$

NOTE. Check each solution by substitution in the original equation.

EXERCISES

Solve for x or n :

- | | | |
|----------------------------|-----------------------|-----------------------|
| 1. $2x = 6a$ | 6. $ax = 2a$ | 11. $2an = 4ab$ |
| 2. $x - a = 2a$ | 7. $ax = 3a^2$ | 12. $\frac{n}{a} = 1$ |
| 3. $x + b = a$ | 8. $2a^2x = 6a^3$ | 13. $\frac{a}{n} = b$ |
| 4. $x - 3b = 5b$ | 9. $cx = d$ | |
| 5. $2x - a = a$ | 10. $x + c = d$ | |
| 14. $cn = c^2 + ac$ | 16. $\frac{n}{p} = r$ | |
| 15. $(c + d)n = (c + d)^2$ | 17. $an + b = b$ | |

Solve for x :

- | | |
|-----------------------------|--------------------------------|
| 18. $5x + 3a = 9x - 12a$ | 23. $ax - a^2 = 5a(a - x)$ |
| 19. $3ax + 5a = ax + a$ | 24. $bx + 5b^2 = 3bx + b^2$ |
| 20. $ax = 8 - 3ax$ | 25. $(a + b)x = 2b + (a - b)x$ |
| 21. $2(x - 5a) = 3(x + 2a)$ | 26. $\frac{1}{2}x = 2m - 1$ |
| 22. $x - b = c - x$ | 27. $\frac{1}{3}x + p = r$ |

Formulas are solved for different letters in the same manner as other literal equations.

For example, the formula for the distance a body passes over in a given time at a given rate is $d = rt$. It is sometimes convenient to express such a formula as solved for r or t in terms of the other letters, as, $r = \frac{d}{t}$ or $t = \frac{d}{r}$.

EXERCISES

- Solve the formula $A = bh$ for b ; for h .
- Find the value of h when $A = 50$ and $b = 5$.
- Find the value of b when $A = 22.5$ and $h = 4.5$.
- Solve the formula $i = prt$ for p ; for r ; for t .
- Find t when $p = \$500$, $i = \$45$, and $r = 4\frac{1}{2}\%$.
- Find r when $p = \$350$, $t = 3$ yr., and $i = \$52.50$.
- Find p when $r = 6\%$, $t = 4$ yr., and $i = \$288$.
- Solve the formula $p = 2l + 2w$ for l ; for w .
- Find l when $p = 87$ rd. and $w = 28.3$ rd.
- If $c = np$, express n in terms of c and p ; express p in terms of c and n .

11. Solve the formula $C = 2\pi r$ for r .

12. (a) Find the circumference of a circle if $r = 11.2$ in.

(Use $\pi = 3\frac{1}{7}$.)

(b) Find the radius of a circle if the circumference is 100 ft.

13. Solve the formula $A = p(1 + rt)$ for p .

14. Using the value of p found in ex. 13, find what sum of money will amount to \$784.80 in 2 yr. at $4\frac{1}{2}\%$.

15. (a) Find a if $l = a + (n - 1)d$.

(b) What is the value of a if $l = 24.5$, $n = 9$, and $d = 2.5$?

16. (a) If $S = \frac{n}{2}(a + l)$, express n in terms of the other letters.

(b) Find n if $S = 510$, $a = 3$, and $l = 57$.

17. (a) If a ton of coal occupies 35 cu. ft. of space, write a formula to represent the number of tons of coal that can be stored in a bin w feet wide, l feet long, and h feet high.

(b) Solve the formula for h .

18. Using the formula of ex. 17, find the height to which 8 tons of coal will fill a bin that is 10 ft. long and 7 ft. wide.

19. Using the formula of ex. 17, find what must be the length of a coal bin that is 5 ft. high and 6 ft. wide if it is to hold 10 tons of coal.

20. (a) Write a formula that states that the cost, c , of sending a telegram is 40¢ for the first 10 words and 3¢ for each additional word.

(b) Solve this formula for n , the number of words the message contains, n being greater than 10.

21. (a) Using the formula of ex. 20, find the cost of sending a message of 15 words.

(b) Find the number of words in a message if the cost of sending it is 94¢.

22. A manufacturer finds that it costs him \$1.75 each to make a certain kind of chair, in addition to overhead charges of \$350. Write a formula to show the total cost, c , of n chairs.

Equations Used in Solving Problems

Many solutions of problems made without the apparent use of equations really imply equations. For example :

(1) Ruth and Mary between them have \$1.50. If Mary has \$.80, how much has Ruth?

The answer is \$.70, because $\$1.50 - \$.80 = \$.70$.

(2) Robert drove several miles into the country and Frank drove 3 times as far. If they drove 20 mi. in all, how far did each drive?

Since Frank drove 3 times as far as Robert did, the total distance was 4 times as far as Robert drove. Therefore Robert drove $\frac{1}{4}$ of 20 mi., or 5 mi., and Frank drove 15 mi.

What is the implied equation in this problem?

More complicated problems may require the actual statement of the equation. When the relations between the numbers are not obvious, you must read the problem carefully to discover the facts that are given and to decide what is required.

As soon as you have decided what is the fundamental unknown number in a problem, represent it by a convenient letter. Then state the other numbers of the problem in terms of the letter that you have selected.

From the conditions stated in the problem, find two expressions that are equal to each other. Equate these expressions. Solve the equation.

EXERCISES

Express the following statements in algebraic language :

1. Twice n is equal to 15.
2. 3 times x increased by 5 equals 23.
3. If 12 is 3 more than n , find the value of n ; of $3n - 2$.
4. If y decreased by 3 equals 8, what is the value of y ? of $2y + 3$?
5. If a is diminished by 5, the result is 13. What is the value of $\frac{1}{2}a + 7$?

6. If n is an odd number, write the next larger even number; the next smaller odd number. Indicate that 3 times the smaller number is 7 more than twice the larger even number.

7. If n is any integer, show that $2n$ must be an even number and that $2n + 1$ must be an odd number.

8. The sum of two numbers is 30. If one of the numbers is n , what is the other number? Indicate that one of the numbers is twice the other number.

9. The difference between two numbers is d . If one of the numbers is 17, what is the other number? (Two answers.) Indicate that the sum of the two numbers is 50.

10. If a boat sails m miles in h hours, how far will it sail in 1 hour? in t hours?

11. If radio tubes cost d dollars a dozen last year and 75¢ less a dozen now, find the per cent of decrease in price.

12. A man sold a pair of skates for a dollars, gaining thereby p per cent of the cost. What was the cost of the skates?

13. If the volume of a room is f cubic feet and the area of the floor is s square feet, what is the height of the room?

14. If a apples cost c cents, how much will n apples cost?

15. Make an equation showing that Edith has d dollars, Mary has twice as much, and the sum of the two amounts is k dollars.

16. Make an equation showing that the cost of t pounds of tea at 50¢ a pound and of $(60 - t)$ pounds at 75¢ a pound is m dollars.

EXAMPLE. If 4 times a certain number diminished by 8 equals twice the sum of the number and 5, what is the number?

HINT.

Let n = the number

$4n - 8$ = four times the number diminished by 8

$2(n + 5)$ = twice the sum of n and 5

Then $4n - 8 = 2(n + 5)$

Solve the equation and get $n = 9$.

Check the result in the original problem.

Problems

1. The sum of two numbers is 49. Twice one of the numbers is 12 less than 3 times the other number. Find the numbers.

HINT.

Let n = one number

Then $49 - n$ = the other number

$2n$ = twice the first number

$3(49 - n)$ = three times the second number

But

$$2n = 3(49 - n) - 12$$

Solve the equation and check the result in the original problem.

2. The sum of two numbers is 24. Three times the first diminished by twice the second is 27. Find the numbers.

3. The sum of two numbers is 51. If twice one of the numbers increased by $\frac{1}{4}$ of the other number is 53, what are the numbers?

4. Five times a certain number exceeds 15 by twice the number increased by 30. What is the number?

5. Find two consecutive numbers such that $\frac{1}{7}$ of the larger number exceeds $\frac{1}{9}$ of the smaller number by 1.

6. One number exceeds a second number by 17. Twice the second number exceeds $\frac{1}{2}$ of the first number by 74. Find the two numbers.

7. Find three consecutive numbers such that $\frac{1}{2}$ of the first plus $\frac{1}{3}$ of the second plus $\frac{1}{4}$ of the third equals the third number.

8. Separate 30 into two parts such that the difference of their squares will be 180.

HINT.

Let p = one part

then $30 - p$ = the other part

p^2 = the square of the first part

$(30 - p)^2$ = the square of the second part

$$\text{But } p^2 - (30 - p)^2 = 180$$

Solve the equation.

9. The sum of two numbers is 25 and the difference between their squares is 125. What are the numbers?

10. The difference between the squares of two consecutive numbers is 39. What are the numbers?

11. John and Arthur together paid \$90 for two radio sets. If one set cost \$26 less than the other, how much did each set cost?

12. The two boys compared the number of stations they could get with their radio sets. The total number of different stations they got was 35. John got 7 stations that Arthur did not get, while Arthur got 13 stations that John did not get. How many stations did each boy get?

HINT. Let n = the number of stations they got in common.

13. As Mr. Thompson was not satisfied with his radio set, he bought another one that enabled him to get 14 more stations than he got with the first set. If on the second set he got 4 stations less than twice the number he got on the first set, how many stations did he get with the new set?

14. The total cost of 3 automobiles bought by three neighbors was \$3040. The first paid \$140 more than the second and \$210 less than the third. How much did each pay?

15. At the end of the year the first two neighbors had driven their cars a total distance of 14,000 mi. How far had each gone if the first drove 50% farther than the second?

HINT. Let n = the number of miles the second drove

Then $n + \frac{50}{100}n$ = the number of miles the first drove

16. Two motor trucks that carry loads of 2 tons and 4 tons respectively were used in moving 58 tons of freight. The smaller truck made 5 trips more than the other truck. How much freight did each truck move?

HINT. Let n = the number of trips the larger truck made

$4n$ = the number of tons the larger truck moved

$n + 5$ = the number of trips the smaller truck made

$2(n + 5)$ = the number of tons the smaller truck moved

Then $4n + 2(n + 5) = 58$, the total number of tons moved
Solve for n . Then find the number of tons each truck moved.

17. A firm used a 3-ton truck and a 4-ton truck to move building material. In moving 82 tons of material, the smaller truck made 4 trips more than the other truck. How much material did each truck move?

18. A freight train contained 3 different types of coal cars. Each car of the first type carried 5 tons less than a car of the second type, and a car of the third type carried 15 tons more than a car of the second type. There were 5 cars of the first type, 7 cars of the second type, and 3 cars of the third type with a total capacity of 845 tons. How much could each type of car carry? (1 ton = 2000 lb.)

19. In a village election, 10% of the qualified voters failed to vote. The winning candidate received 25 votes more than $\frac{1}{2}$ of the number of votes cast, while the losing candidate received 74 votes. How many votes were cast? How many qualified voters were there?

20. Aleck has 29¢ and Robert has 15¢. How much should Aleck give Robert so that they may have equal amounts?

21. Dora had \$39 and Ruth had \$12. Dora gave Ruth some money and Ruth then had $\frac{1}{2}$ as much as Dora. How much did Dora give Ruth?

22. Divide \$51.50 among 5 men and 3 boys so that each man shall receive \$4.70 more than each boy.

23. Mr. Chase received annual increases of 20% of his salary. If he earned \$3640 the first 3 yr., what was his salary each year?

24. The perimeter of one square is 16 in. greater than that of another square and its area is 48 sq. in. greater. Find the length of a side of each square.

25. The sum of the areas of two squares is 130 sq. in. and the difference of their areas is 32 sq. in. What is the area of each square? What is the length of the side of each square?

26. A rectangle has the same area as a square. The length of the rectangle is 8 in. greater and the width is 4 in. less than the side of the square. Find the dimensions of each figure.

Problems on uniform motion. Actual uniform motion is seldom found. The moving body, whether it is a steam locomotive, an automobile, a bullet from a rifle, or some other moving body, goes at varying speeds; but for practical purposes in the problems that are given, the average speed is taken as being uniform, that is, at the same rate throughout the time considered.

Oral Problems

1. How far will Jane walk in 4 hr. at a rate of 3 mi. an hour? at a rate of r miles an hour?

2. If Fred rides his bicycle at a rate of 8 mi. an hour, how far will he go in 2 hr.? in 5 hr.? in t hours?

3. How long will it take to travel 12 mi. at a rate of 3 mi. an hour? at 6 mi. an hour? at r miles an hour?

4. Give a formula to express the distance, d , that will be passed over by a car running at r miles an hour for t hours.

5. An automobile went m miles at a rate of r miles an hour. How long did it take?

6. John and Richard started at the same time from the same place and walked in opposite directions. John walked 4 mi. an hour and Richard walked 3 mi. an hour. How far apart were they at the end of 1 hr.? of 2 hr.? of h hours?

7. If John and Richard walked in the same direction, how far apart were they in 1 hr.? in 2 hr.? in h hours?

8. If John and Richard were 21 mi. apart and started at the same time to walk toward each other, at what rate did they approach each other? How far apart were they after 1 hr.? How long was it before they met?

9. A power boat goes 12 mi. an hour in still water. How fast can it go up a river if the current of the water is 2 mi. an hour? What will be the speed of the boat down the river?

10. If the rate of the boat in still water is b miles an hour and the rate of the stream is r miles an hour, express the rate of the boat upstream; downstream.

Sound travels through the air at about 1100 ft. a second. Light travels at about 186,000 mi. a second. Hence for practical distances light is instantaneous.

11. Jerry saw the steam from a boat's whistle and 3 sec. later heard the whistle. How far away was the boat?

12. If thunder was heard 5 sec. after the lightning flash was seen, how far away was the electric discharge?

Problems

In ex. 1 to 5, two drivers start at the same place and time and travel in opposite directions.

1. The first goes 12 mi. an hour and the second, 16 mi. an hour. How long will it take them to be 126 mi. apart?

2. The first travels twice as fast as the second. How fast does each go if they are 114 mi. apart in 4 hr.?

3. The first goes 3 mi. an hour faster than the second. In 4 hr. they are 188 mi. apart. What is the rate of each?

4. The first goes at a rate of 30 mi. an hour and the second, 12 mi. an hour faster. How long will it be before they are 108 mi. apart? 180 mi. apart?

5. The first goes 4 mi. an hour faster than the second. How fast does each travel if they are 210 mi. apart in $3\frac{1}{2}$ hr.?

6. Two drivers start from Denver. The first drives north toward Greeley. The second starts south toward Colorado Springs half an hour later at 5 mi. an hour faster than the first. At what rate does each travel if they are $133\frac{1}{2}$ mi. apart 3 hr. after the first driver has left?

7. Mr. Smith leaves on a trip in his automobile at a rate of 24 mi. an hour. His son starts 2 hr. later to overtake him. At what rate must his son drive to overtake him in 6 hr.?

8. Henry and David take a trip on their motorcycles. Henry has tire trouble and is delayed 20 min., but David continues, riding along at 20 mi. an hour. At what rate does Henry travel if it takes him 50 min. to overtake David?

9. A freight train starts west from Chicago at a rate of 20 mi. an hour. If an express train running in the same direction at a rate of 44 mi. an hour leaves $2\frac{1}{2}$ hr. later, how far will it go before it overtakes the freight train?

10. A train from New York to Buffalo by a route over the mountains makes the trip in 10 hr. Another train by a water-level route 41 mi. longer makes the trip in 1 hr. less time. Find the rate of each train if the second train runs 9 mi. an hour faster than the first train. Find each distance.

11. Philadelphia and Washington are 143 mi. apart. At 1 P.M. an automobile leaves Washington for Philadelphia at a speed of 24 mi. an hour. After running 1 hr. it is delayed 40 min. At 2 P.M. another automobile leaves Philadelphia for Washington at a speed of 21 mi. an hour. At what time will the two cars meet? How far will each have gone?

12. Ida takes $3\frac{1}{2}$ hr. to get to a house by electric cars. She returns in 2 hr. 20 min. by automobile at a speed of 9 mi. an hour faster than she goes. Find both rates of speed and the distance.

13. A man having 8 hr. at his disposal rides by automobile at a rate of 24 mi. an hour to the Zoölogical Park. He spends $4\frac{1}{4}$ hr. at the park and walks back at the rate of 3 mi. an hour. How far does he ride?

14. Louis can row 4 mi. an hour in still water. He finds that he can row up a river the same distance in 3 hr. that he can row down the river in $1\frac{1}{2}$ hr. Find the rate of the current of the river.

15. Some boys hire a launch that runs at a rate of 10 mi. an hour in still water. How far can they go up a river and return in 5 hr. if the rate of the current of the river is 3 mi. an hour?

16. Three seconds after the report a marksman hears the bullet hit a target that is 2100 ft. distant. If sound travels 1100 ft. a second, what is the average velocity of the bullet?

Coin problems, etc.

1. A collection of 18 nickels and dimes amounted to \$1.15. How many coins were there of each?

HINT.

Let d = the number of dimes

Then $18 - d$ = the number of nickels

Since each dime is worth 10 cents, d dimes = $10d$ cents

and $(18 - d)$ nickels = $5(18 - d)$ cents

$$\text{Therefore } 10d + 5(18 - d) = 115$$

Solve the equation for d .

2. A purse contained \$.60 in cents and nickels. How many were there of each if there were 20 coins?

3. After selling papers, Arthur found that he had 35 coins in his pocket, consisting of quarters, dimes, and nickels. How much money did he have if there were 4 more dimes than nickels and $\frac{1}{4}$ as many quarters as other coins?

4. Paul had 3 times as many nickels as dimes. Fred had as many nickels as Paul had dimes and as many dimes as Paul had nickels. Together they had \$3. How much did each boy have?

5. Edward had \$1.47 in quarters, nickels, and cents. He had twice as many nickels as quarters and the number of cents was 1 less than the number of nickels. How many did he have of each?

6. Julia spent 50¢ for 1-cent and 2-cent postage stamps. How many did she buy of each if the number of 2-cent stamps was 7 more than the number of 1-cent stamps?

7. Mr. Clark sent his son for \$1 worth of 1-cent, 2-cent, and 5-cent postage stamps. The boy returned with 10 more 1-cent stamps than 5-cent stamps, and twice as many 2-cent stamps as 1-cent stamps. How many stamps of each kind did he buy?

8. The total receipts from the sale of 475 tickets for a school entertainment were \$168.75. If adults paid \$.50 each and children \$.25, how many adults bought tickets? how many children?

Mixture problems. The grocer, the druggist, and the chemist are constantly making mixtures. Some of the simpler problems which they encounter will be considered in this section.

1. A merchant sold a blend of 35-cent and 50-cent coffees at 44¢ a pound. How many pounds of each kind of coffee were there in a 50-pound mixture?

HINT.

Let n = the number of pounds of 35-cent coffee
 and $50 - n$ = the number of pounds of 50-cent coffee
 Then $35n$ = the value in cents of the 35-cent coffee
 and $50(50 - n)$ = the value in cents of the 50-cent coffee

Then $35n + 50(50 - n) = 2200$, the value of the whole mixture
 Solve the equation for n .

2. How many pounds of 40-cent coffee should a grocer mix with 20 pounds of 52-cent coffee so that the mixture may be sold at 48¢ a pound?

3. A grocer wished to make a mixture of coffee sold at 42¢ a pound with coffee sold at 50¢ a pound. How many pounds of each grade of coffee should be put in a 48-pound mixture to be sold at 45¢ a pound?

4. A tea merchant sold a mixed tea at 75¢ a pound. He put 24 lb. of tea at 70¢ a pound with some tea at 90¢ a pound. How many pounds of the dearer grade of tea were there in the mixture?

HINT. $70 \times 24 + 90p = 75(24 + p)$. Explain.

5. The price of milk was 18¢ a quart and the price of cream was 75¢ a quart. As some customers desired a richer milk, the dealer mixed cream with 15 qt. of milk. He sold the mixture at 30¢ a quart. How many quarts of cream did he add?

6. How many pounds of Brazil nuts selling at 32¢ a pound should a dealer add to 25 lb. of walnuts selling at 45¢ a pound so that the mixture may sell at 40¢ a pound?

Age problems.

1. The sum of the ages of Mary and John is 32 yr. In 6 yr. from now Mary will be 5 times as old as John was 8 yr. ago. How old is each now?

HINT. Let y = the number of years in Mary's age now
 Then $32 - y$ = the number of years in John's age
 $y + 6$ = the number of years in Mary's age 6 yr. hence
 $(32 - y) - 8$ = the number of years in John's age 8 yr. ago

Therefore $y + 6 = 5(32 - y - 8)$, or $y + 6 = 5(24 - y)$
 Solve the equation for y .

2. The sum of the ages of Eva and Agnes is 23 yr. In 8 yr. Eva will be twice as old as Agnes. How old is each now?

3. Ruth is 24 yr. younger than her mother. In 3 yr. her mother will be 7 times as old as Ruth was 3 yr. ago. How old is each now?

4. How old is Henry if in 7 yr. he will be twice as old as he was 3 yr. ago?

5. Mr. Bruce is 4 times as old as his son; but 6 yr. ago he was 10 times as old as his son. How old is each?

6. Allen's teacher is 5 yr. less than 3 times as old as Allen; but 7 yr. ago the teacher was twice as old as Allen will be next year. How old is each?

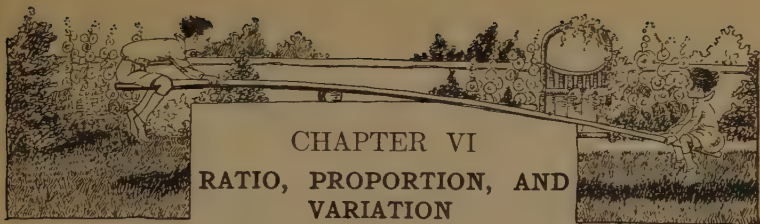
7. Frank is 3 yr. younger than Rob. In 6 yr. he will be twice as old as Rob was 6 yr. ago. Find the age of each.

8. How old is John if $\frac{1}{3}$ of his age 5 yr. ago equals $\frac{1}{5}$ of his age 5 yr. hence?

9. Walter is $\frac{3}{5}$ as old as Henry, but 6 yr. hence Walter will be only $\frac{3}{4}$ as old as Henry. How old is each?

10. Ruth is 1 yr. older than Grace, but 3 yr. ago Grace was $\frac{2}{3}$ as old as Ruth will be in 2 yr. from now. How old is each?

11. Frank is $\frac{5}{12}$ as old as his father; but 8 yr. ago he was $\frac{1}{4}$ as old as his father was then. How old is each?



Ratio

If you work 10 exercises of a set and do 9 of them correctly, you may say that you do $\frac{9}{10}$ of them correctly. In other words, the relation of 9 to 10 is $\frac{9}{10}$. The word *ratio* means *relation*. Hence you may say that the ratio of the number you do correctly to the entire number you try is 9 to 10, or $\frac{9}{10}$.

The **ratio** of one number to another number of the *same kind* is the indicated quotient of the first number divided by the second number.

Ratios are generally expressed as fractions, the first number being the numerator of the fraction and the second number, the denominator; but they may also be written with a colon between the two numbers.

Thus, the ratio of 3 to 5 may be written $\frac{3}{5}$ or 3 : 5. The ratio of a to b may be written $\frac{a}{b}$ or $a : b$.

The terms of a ratio must be numbers of the same kind.

Since ratios are fractions, any operations that are permissible with fractions are permissible also with ratios. Ratios should always be reduced to the simplest form.

Thus, the ratio of $\frac{6}{12}$ should be reduced to $\frac{1}{2}$.

EXERCISES

Express each of the following ratios in its simplest form :

1. $\frac{4}{12}$

3. $\frac{24 a}{36 a}$

5. $\frac{3.6}{7.2}$

7. $\frac{65 \text{ ft.}}{39 \text{ ft.}}$

2. $\frac{6}{18}$

4. $\frac{21 b^2}{35 b}$

6. $\frac{4.5}{.9}$

8. $\frac{2\frac{1}{2} \text{ tons}}{7\frac{1}{2} \text{ tons}}$

9. What is the ratio of 2 ft. to 1 yd.? of 5 in. to 1 ft.?

10. What is the ratio of 2 lb. 7 oz. to 3 lb. 4 oz.?

11. What is the ratio of 10 rd. to 1 mi.?

12. John earns \$24 a week and spends \$15 a week for his room rent and meals. What is the ratio of this expense to his wages? of his wages to his rent and board bill?

13. Mary does correctly 27 exercises out of 36. What is the ratio of her correct exercises to the total?

14. In a class of 36 boys and girls there are 20 girls. What is the ratio of girls to the whole class? of boys to girls? of girls to boys?

15. In a freight train of 57 cars there are 38 coal cars. What is the ratio of coal cars to all the cars?

16. Divide \$100 between two men so that their shares shall be as 3 is to 7.

HINT. The first man will receive 3 parts, or $3p$.

The second man will receive 7 parts, or $7p$. Then $3p + 7p = 100$.

17. A business capitalized at \$16,000 is owned by two men whose shares are as 3 is to 5. If the profits at the end of the year are 24% of the capitalization, how much money does each partner receive?

18. In a class of 32 pupils the ratio of boys to girls is 5 to 3. How many boys are there? how many girls?

19. Two boys together weigh 230 lb. If the ratio of their weights is 11 to 12, how much does each boy weigh?

20. Find each angle of a triangle if they are in the ratio of 1 to 2 to 3; if they are in the ratio of 1 to 2 to 2.

Proportion

If the two sides of an equation are fractions, as $\frac{2}{3} = \frac{4}{x}$ or $\frac{a}{b} = \frac{c}{d}$, the equation is called a *proportion*.

A **proportion** is a statement of equality between two equal ratios. Proportions may be written thus:

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad a:b = c:d$$

The proportion $\frac{a}{b} = \frac{c}{d}$ is read “*a is to b as c is to d.*”

The first and last terms of a proportion (as *a* and *d* above) are called the **extremes**. The two middle terms (as *b* and *c* above) are called the **means**.

If four numbers are given as being in proportion, it is understood that they are to be taken in the order named. Thus, if 6, 9, 8, and 12 are given, the proportion is $\frac{6}{9} = \frac{8}{12}$. Similarly, if the numbers are *a*, *b*, *c*, and *d*, the proportion is $\frac{a}{b} = \frac{c}{d}$.

The proportion $\frac{6}{9} = \frac{8}{12}$ may be written as the equality of two products if each side is multiplied by 9×12 . Thus, $6 \times 12 = 9 \times 8$. Similarly, the proportion $\frac{a}{b} = \frac{c}{d}$ becomes $ad = bc$. These products show an important fundamental law of a proportion, namely:

In any proportion the product of the extremes equals the product of the means.

This property enables one to determine easily whether four numbers form a proportion.

EXAMPLE 1. Do the four numbers 6, 10, $4\frac{1}{2}$, and $7\frac{1}{2}$ form a proportion?

SOLUTION. $6 \times 7\frac{1}{2} = 45$; $10 \times 4\frac{1}{2} = 45$. Hence the proportion $6 : 10 = 4\frac{1}{2} : 7\frac{1}{2}$ is a true proportion.

EXAMPLE 2. What is the value of x in $\frac{2}{3} = \frac{6}{x}$?

SOLUTION.

$$\frac{2}{3} = \frac{6}{x}$$

Find the product of the extremes and of the means: $2x = 18$

$$x = 9$$

CHECK. $\frac{2}{3} = \frac{6}{9}$

EXERCISES

Find the value of x and check:

1. $\frac{3}{5} = \frac{9}{x}$

4. $\frac{6}{15} = \frac{x}{7.5}$

7. $21:49 = 28:3x$

2. $\frac{x}{15} = \frac{8}{5}$

5. $\frac{24}{28} = \frac{6}{x}$

8. $25:2x = 16:40$

3. $\frac{3}{7} = \frac{x}{21}$

6. $\frac{5x}{12} = \frac{25}{18}$

9. $21:15 = 7x:20$

10. $12:27 = 8:9x$

11. Solve the proportion $\frac{4}{6} = \frac{4-x}{x}$ for x .

SOLUTION.

$$\frac{4}{6} = \frac{4-x}{x}$$

Find the product of the extremes and of the means: $4x = 24 - 6x$

Solve for x :

$$x = 2.4$$

CHECK. If $x = 2.4$, $\frac{4}{6} = \frac{4-2.4}{2.4}$, or $\frac{4}{6} = \frac{1.6}{2.4}$, or $\frac{2}{3} = \frac{2}{3}$

Solve for x :

12. $\frac{x}{x+2} = \frac{6}{10}$

17. $\frac{6}{x+4} = \frac{3}{x+1}$

13. $\frac{x-1}{x} = \frac{28}{35}$

18. $\frac{x+3}{15} = \frac{x-3}{6}$

14. $\frac{3+x}{x} = \frac{19}{18}$

19. $\frac{2x-5}{x+3} = \frac{2.5}{15}$

15. $\frac{18}{15-x} = \frac{15}{x}$

20. $\frac{x-7}{2x+4} = \frac{5}{1}$

16. $\frac{18-x}{9+x} = \frac{5}{4}$

21. $\frac{3x+3}{7x-8} = \frac{2}{3}$

22. Determine whether the following numbers form a proportion:

(a) 4, 3, 16, 12

(b) 7, 5, 11, 8

(c) 24, 18, 4, 3

23. In $3 \times 8 = 4 \times 6$, show that either pair of factors may be the extremes and the other pair the means of a proportion.

SOLUTION.

$$3 \times 8 = 4 \times 6$$

Divide both sides by 4×8 : $\frac{3 \times 8}{4 \times 8} = \frac{4 \times 6}{4 \times 8}$, or $\frac{3}{4} = \frac{6}{8}$

In a similar way, $\frac{4 \times 6}{4 \times 8} = \frac{3 \times 8}{4 \times 8}$

Divide both sides by 3×6 : $\frac{4 \times 6}{3 \times 6} = \frac{3 \times 8}{3 \times 6}$, or $\frac{4}{3} = \frac{8}{6}$

24. Form two proportions with 4 as the first term from the identity $4 \times 18 = 6 \times 12$.

25. Form two proportions with a as the first term from the equation $ax = by$.

Find the ratio between x and y if:

26. $5x = 3y$

28. $8x = 4y$

30. $3x = y$

27. $3x = 7y$

29. $12y = 9x$

31. $15y = 27x$

32. Two numbers are in the ratio of 3 to 5. If 4 is subtracted from the smaller number and 8 is added to the larger number, the ratio becomes 5 to 12. Find the numbers.

SOLUTION. Let the two numbers be $3n$ and $5n$.

Then $\frac{3n - 4}{5n + 8} = \frac{5}{12}$

Solve the equation by taking the product of the means and of the extremes:

$$12(3n - 4) = 5(5n + 8)$$

Solve: $n = 8$

$$3n = 24 \text{ and } 5n = 40$$

CHECK. $\frac{24}{40} = \frac{3}{5}$ and $\frac{24 - 4}{40 + 8} = \frac{20}{48} = \frac{5}{12}$

33. Two numbers are in the ratio of 3 to 4. If 7 is subtracted from each number, the remainders are in the ratio of 2 to 3. Find the numbers.

34. Two numbers are in the ratio of 4 to 5. If each is increased by 5, the resulting numbers are in the ratio of 5 to 6. Find the original numbers.

35. The numerator of a certain fraction is to its denominator in the ratio $\frac{2}{3}$. If 5 is added to the numerator, the ratio is $\frac{3}{2}$. What is the fraction?

36. Separate 45 into two parts having the ratio of 2 to 3.

HINT. See ex. 16, page 98, or use $n : 45 - n = 2 : 3$. Explain.

37. Divide 63 into two parts having the ratio of 3 to 4.

38. Divide 75 into three parts proportional to 4, 5, and 6.

39. John is 15 yr. old and Henry is 18 yr. old. Divide \$38.50 between John and Henry in proportion to their ages.

40. The ratio of boys to girls in an algebra class of 42 is 4 to 3. How many boys must withdraw so that there shall be 6 girls for every 7 boys?

41. The perimeter of a certain parallelogram is 56 in. Find the sides of the parallelogram if they are in the ratio of 2 to 5.

42. It requires 154 rd. of fencing for a rectangular field. What are the dimensions of the field if they are in the ratio 3 : 8?

43. One side of a rectangular playground is 25 yd. longer than the other side. If the sides are as 12 is to 17, find the dimensions of the playground.

44. The distance around a triangular grass plot is 144 ft. Find the sides if they are in the ratio 4 : 5 : 7.

45. Find the angles of a triangle if they are in the ratio 1 : 1 : 2.

46. Find the angles of a triangle if they are in the ratio 1 : 3 : 5.

47. The numerator of a certain fraction is to the denominator as 9 to 16. Find the fraction if the denominator is 32 less than twice the numerator.

Functions

The mark that you receive on an algebra test depends on the number of the given exercises that you do correctly. If you take a trip on a train, the distance you go depends on the rate of the train and on the length of time you travel.

When one quantity y depends on another quantity x for its value, y is said to be a **function** of x . This is written $y = f(x)$.

In the equation $y = 2x - 3$, $y = f(x)$ or $f(x) = 2x - 3$. In this case, if x increases, $f(x)$ increases, while if x decreases, $f(x)$ decreases.

ORAL EXERCISES

Complete the statements containing blanks:

1. When you go home from school, on what does the time you take depend? The time is a function of —.
2. If money is deposited in a savings bank the amount of interest depends on —, —, and —.
3. If you buy potatoes, on what does the amount of money you pay depend? The cost is a function of —.
4. On what does the circumference of a circle depend? The circumference of a circle is a function of —.
5. The area of a rectangle whose altitude is 6 in. is a function of —.
6. The perimeter of a parallelogram is a function of —.
7. If $n = 2x$, at what rate does n increase as x increases?
8. If $n = \frac{x}{y}$ does n increase if x increases? if y increases?
9. If $f(x) = 3x + \frac{2}{x}$, does $f(x)$ increase if x increases?
10. If $n = \frac{2x - 3}{y + 2}$, does n increase if x increases? Does n increase if y increases?
11. In ex. 10, what change occurs in n if x decreases? if y decreases?

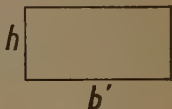
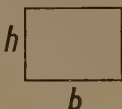
EXERCISES

1. If an automobile goes 60 mi. in 2 hr., how far will it go at the same rate in 4 hr.? in 3 hr.? in $\frac{1}{2}$ hr.?

2. If 2 pairs of shoes cost \$15, at the same rate how much will 4 pairs cost? 5 pairs? 8 pairs?

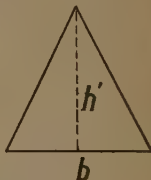
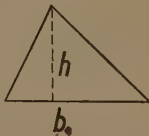
3. The circumference of a circle is 60 in. What is the circumference of a circle whose diameter is twice as great? 5 times as great? $\frac{1}{2}$ as great?

4. The area of a rectangle having a constant altitude varies as its base. Show that this is true from the formula for the area. Measure the bases of the two rectangles at the right. How do they compare? How do the areas compare?



NOTE. b' is read " b prime." It is different from the b first used.

5. The area of a triangle having a constant base is a function of its height. Show that this is true from the formula for the area of a triangle. Measure the heights of the two triangles in the figure at the right. Compare the areas of the triangles.



6. If the area of a triangle is 16 sq. ft., what will be the area of a triangle having the same base but a height 3 times as large? a height $\frac{1}{4}$ as large?

7. Compare the area of a parallelogram with the area of another parallelogram having the same height but a base 3 times as large.

8. Map drawing is done to a scale. If 1 in. on a map represents 10 mi. on the earth's surface, what is the actual distance between two places that are $3\frac{1}{2}$ in. apart on the map? If two towns are 18 mi. apart, what should be the distance between them on such a map?

Variation

If a man drives his automobile at a rate of 25 mi. an hour, the distance he goes depends on the number of hours he drives; that is, $d = 25h$. Such dependence is called **variation**. We say that the distance *varies directly* as the time. If the time is increased, the distance is increased.

Thus, in 2 hr. the distance will be 50 mi. In twice 2 hr., or 4 hr., the distance will be twice 50 mi., or 100 mi.

If another man drives at the same rate for a different number of hours, the distance he goes is $d' = 25h'$. In comparing the two distances it is stated that $\frac{d}{d'} = \frac{25h}{25h'}$, or $\frac{d}{d'} = \frac{h}{h'}$. That is, the distance a man travels at a given rate is *directly proportional* to the time he travels.

A **variable number** or **quantity** is one whose value continues to change throughout a discussion.

A **constant** is a number that has the same value throughout a discussion. That is, it does not change.

Thus, in the expression $d = 25h$, d and h are variables, since they change in value; but 25 is a constant.

When one number varies directly as another number, the ratio between any two corresponding values of the numbers is a *constant number*; that is, the ratio is fixed.

Thus, if $d = 25h$, $\frac{d}{h} = 25$. Similarly, $C = \pi d$, or $\frac{C}{d} = \pi$; that is, the circumference of a circle varies as the diameter.

In general, if one number a varies directly as another number n and if k represents some constant, $a = kn$.

Complete the following statements:

1. The area of a triangle with a constant base varies as —.
2. The area of a parallelogram with a constant altitude varies as —.

Inverse variation. The area of a rectangle is equal to the product of its base by its height. The equation $bh = b'h'$ indicates that two rectangles with dimensions b, h and b', h' , respectively, are equal. Dividing both sides of the equation by $b'h$, we get the proportion $\frac{b}{b'} = \frac{h'}{h}$. That is, the bases are inversely proportional to the heights.

Thus, the area of a rectangle with height 6 and base 8 is 48. If the height of this rectangle is doubled and becomes 12, while the area remains the same, 48, the new base is $\frac{1}{2}$ of the old base, or 4. In the same way, if the height is decreased to 4, the base is increased to 12.

If one quantity decreases as a second quantity increases, the two quantities are said to vary inversely.

EXERCISES

1. If 4 men can build a barn in 12 days, how long will it take 6 men to build the same barn?

HINT. This is an illustration of inverse variation. If the number of men is increased, the number of days required to do the work will be decreased. The problem may be explained as follows: The work of 4 men for 12 days is equivalent to 48 work days, which is equivalent to the work of 6 men for n days. Hence we have the equation $4 \times 12 = 6n$ or the proportion $4 : 6 = n : 12$.

2. Four boys dug a trench in 6 hr. How long would it have taken 5 boys?

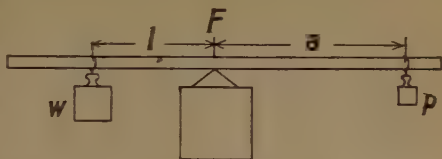
3. If there is on hand enough oats to feed 6 horses 6 days, how long will the oats last 4 horses?

4. If the oats will feed h horses d days, how long will they feed n horses?

5. The dimensions of a parallelogram are base 9 in. and height 6 in. What is the height of an equivalent parallelogram whose base is 18 in.?

6. The time that it takes to go a fixed distance varies inversely as the rate of travel. Express this fact by a proportion.

The **lever** is a bar balanced at some point, as F , that is called the **fulcrum**. If the fulcrum is in the middle of the bar, equal weights at the ends will balance the lever; but if the fulcrum is not in the middle, a larger weight on the shorter arm is necessary to bal-



ance a weight at the other end. The common teeter or seesaw is a good illustration of the lever (see page 14). Wrecking bars and claws used for prying out nails, etc. are levers.

If a lever is balanced, the weights w and p are inversely proportional to the lengths of their arms, l and a .

$$\frac{w}{p} = \frac{a}{l}$$

EXERCISES

1. A lever is 20 ft. long and the fulcrum is 8 ft. from the weight. If the weight is 160 lb., what power is needed at the other end to just balance the bar?

2. Two boys are playing seesaw on a board that is 18 ft. long. How far from the heavier boy should the point of support be placed if he weighs 90 lb. and the other boy weighs 75 lb.?

3. An 8-ounce weight is attached to one end of scales 2 in. from the support. How great is a weight that just balances the scales when placed 4 in. from the fulcrum? 6 in. from the fulcrum? 12 in.?

4. A man wishes to move a rock weighing 200 lb. with a lever that is 12 ft. long. If the fulcrum is placed 1 ft. from the rock, how great a pressure must the man exert if he is 6 ft. from the fulcrum? 8 ft. from the fulcrum? at the other end of the lever? What would be the effect if the fulcrum were placed 6 in. from the rock? 2 ft. from the rock?

EXERCISES

1. The taxes on property assessed for \$6800 are \$136. At the same rate, what would be the taxes on property assessed for \$7200?

2. The interest on \$4800 for $1\frac{1}{2}$ yr. is \$324. What will be the interest on \$6000 for the same time at the same rate?

3. The area of a circle is proportional to the square of its radius. (a) State this relation in the form of a proportion.

(b) If the area of a circle is 100 sq. in., what is the area of a circle that has a radius twice as large? 3 times as large?

4. If the temperature remains constant, the volume of a body of gas varies inversely as the pressure. (a) State this as a proportion.

(b) The volume of a body of gas under a pressure of 15 lb. per square inch is 18 cu. ft. What will be the volume if the pressure is 24 lb. per square inch? 12 lb. per square inch?

5. Three quarts of alcohol are mixed with 5 qt. of water. How many quarts of alcohol must be added so that the mixture will be $\frac{3}{4}$ alcohol?

SOLUTION. The mixture already contains $\frac{3}{8}$ alcohol. (Why?)

Let n = the number of quarts of alcohol to be added

$$\text{Then } \frac{3+n}{8+n} = \frac{3}{4}$$

Solve:

$$n = 12$$

CHECK. The new mixture will contain 20 qt., of which 15 qt. are alcohol.
 $15 : 20 = 3 : 4$.

6. A salt-water solution contains 1 oz. of salt and 23 oz. of water. How much salt must be added to make it a 10% solution?

7. How much water must be added to a gallon of alcohol 80% pure to make it 60% pure?

8. A certain baseball team wins 34 games more than it loses. At the close of the season its standing is .625; that is, the ratio of the number of games it has won to the number it has played is $62\frac{1}{2}\%$. How many games has it played?

Geometric Exercises in Proportion

1. If a line DE is drawn parallel to the base BC of a triangle, the sides are divided proportionally. That is, $AD : DB = AE : EC$, or $m : n = p : q$. Find q if $m = 5$, $n = 3$, and $p = 7\frac{1}{2}$.

2. In the same figure, find m if $n = 8$, $p = 16$, and $q = 5$.

3. In the same figure, find AC if $AD = 12$, $DB = 15$, and $AE = 9$.

4. In the triangle below, if ED is drawn parallel to CB , $AD : ED = AB : CB$. If $AD = 5$, $ED = 4$, $AB = 15$, find CB .

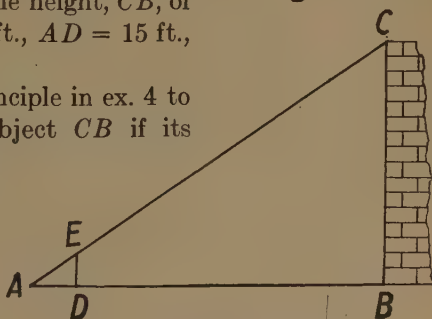
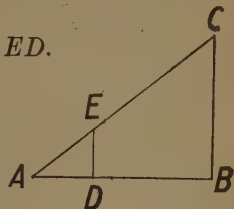
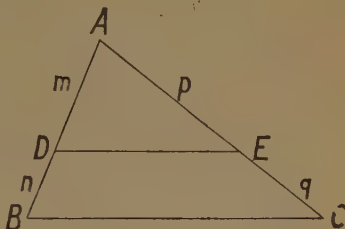
5. If $AD = 3$, $AB = 18$, $CB = 28$, find ED .

6. The principle illustrated in the figure for ex. 4 may be used to find the heights of trees or other high objects. At point A on the ground E and C are sighted in a straight line. Find the height, CB , of a building if $AB = 112.5$ ft., $AD = 15$ ft., and $ED = 10$ ft.

7. Make use of the principle in ex. 4 to find the height of an object CB if its shadow AB is 80 ft. long at the same time that a post ED 4 ft. high casts a shadow 5 ft. long.

8. Find the height of a tree that casts a shadow 100 ft. long when a fence post 6 ft. tall casts a shadow $4\frac{1}{2}$ ft. long.

9. A pole for the antenna of a radio set casts a shadow 25 ft. long when a man 5 ft. 10 in. tall casts a shadow 7 ft. long. What is the height of the pole?

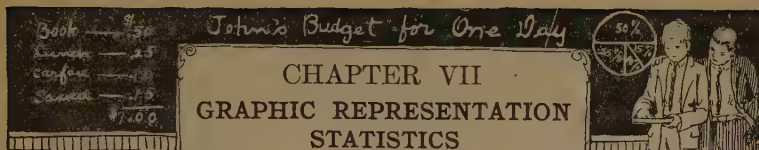




Fun with Figures



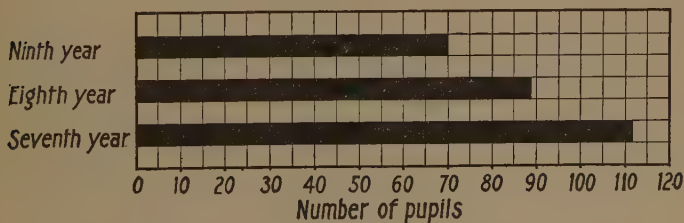
1. Write a number of two digits that is twice the sum of its digits.
2. If eggs are 60¢ a dozen, what is the difference in price of 6 dozen dozen eggs and half a dozen dozen eggs?
3. Matilda had 30 apples in one basket that she sold for 3 for 5¢, and 30 apples in another basket that she sold for 2 for 5¢. When all the apples were sold she had \$1.25. The next day her brother Fred took 60 apples to market and sold them at the rate of 5 for 10¢, but he took in only \$1.20. Explain the difference.
4. How may 51 qt. of cranberries be divided into three equal portions by using only a 4-quart and a 7-quart measure?
5. In how many ways can 5 people be seated around a table?
6. A man bought a radio set for \$17.50 and gave a \$20 bill in payment. As the dealer could not change the bill, he had it changed next door and gave the purchaser his change. Later it was discovered that the bill was counterfeit and the dealer replaced it. How much did the dealer lose?
7. Write any number of three digits, as 852; then write the number formed by reversing the digits, as 258. The difference between the two numbers (in this case, 594) is always divisible by 9, by 11, and by the difference between the first and the last digits. Explain.
8. Divide \$17 among 3 girls so that one may have $\frac{1}{2}$ of the money, another $\frac{1}{3}$ of the money, and the third $\frac{1}{6}$ of the money.
9. The first of two bottles of the same size is half full of water and the second is half full of ginger ale. From the first bottle a quarter of a cup of water is taken and poured into the second bottle. From the second bottle a quarter of a cup of the mixture is taken and put into the first bottle. Is the quantity of water removed from the first bottle greater or less than the amount of ginger ale taken from the second bottle?



In the last chapter you studied the relation between numbers expressed in the form of ratio and proportion. In this chapter you will study the graphic method of indicating number relations, which gives a more vivid impression of the relations between numbers.

Bar Graphs

Suppose, for example, that you wish to show that the pupils in a Junior High School were distributed as follows: seventh year, 112; eighth year, 88; ninth year, 70. The relations of these three classes to each other may be seen at a glance if they are pictured by three bars, as shown below. Such a comparison of numbers in pictured form is called a **graph**.



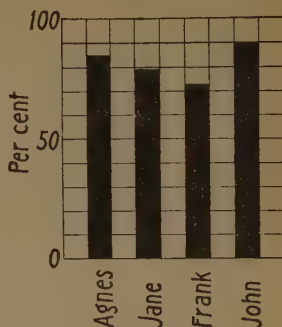
DISTRIBUTION OF THE PUPILS IN A JUNIOR HIGH SCHOOL

The bars pictured above are constructed on squared paper called **cross-section paper**, or **coördinate paper**. Their length is determined by the *scale* of the graph. In this graph the scale is 5 pupils for each square. Squared paper is not necessary for the construction of a bar graph; but it is a convenient aid.

You may have learned before how to draw and interpret various kinds of graphs; but since your textbooks, as well as current trade journals, newspapers, and other periodicals, frequently present numerical facts in the form of graphs, it is important that you should review and extend your knowledge of them.

The numerical facts that are given constitute the *statistical data* upon which the graphs are based.

From the vertical bar graph at the right, read the mark of each pupil.



THE MARKS IN ALGEBRA
RECEIVED BY FOUR PUPILS

The following directions for making a bar graph should be observed.

1. *Before making the graph, the scale to be used should be determined.*

In the graph on page 111, each horizontal space represents 5 pupils. In the graph above, each vertical space represents an average of 10%. If the data to be graphed run into large numbers, each space may represent 100, 1000, or even more.

2. *The zero of the scale must be given.*

Otherwise the bars would not accurately compare the data given. For example, in the graph above, if the base line were given at 50, the bars for Frank and John would indicate that John's mark was nearly twice as great as Frank's mark.

3. *The bars should be of uniform width throughout and the spaces between the bars should be of the same width.*

4. *Each bar should be carefully labeled.*

5. *The title of the entire graph should be clearly stated beneath it.*

6. *All the lines and the lettering should be neatly and accurately drawn.*

EXERCISES

Construct bar graphs to represent the following facts:

1. The number of boys at a school who reported for certain athletic teams was as follows: football, 50; baseball, 45; track, 70; basket ball, 30.

2. The average marks obtained by the girls and boys of a school were: seventh year — girls, 80; boys, 75; eighth year — girls, 82; boys, 78; ninth year — girls, 77; boys, 88.

3. The growth of Chicago from 1870 to 1920 is shown by the following census figures, given to the nearest 100,000:

1870 — 300,000	1890 — 1,100,000	1910 — 2,200,000
1880 — 500,000	1900 — 1,700,000	1920 — 2,700,000

4. The average prices per bushel of grains one year were:

Wheat, \$1.62	Oats, \$.54	Rye, \$1.26
Corn, \$1.12	Barley, \$.82	Buckwheat, \$1.07

5. The football records of four colleges one season were:

COLLEGE	GAMES PLAYED	GAMES WON	PER CENT WON
A	7	5	71
B	8	5	63
C	8	4	50
D	9	7	78

Make a graph of the percentage of games won. Why would it not be fair to make a graph of the number of games won?

6. The number of passenger automobiles made in the United States from 1921 to 1926, to the nearest 100,000, were:

1921 — 1,400,000	1923 — 3,500,000	1925 — 3,700,000
1922 — 2,400,000	1924 — 3,200,000	1926 — 3,900,000

7. The retail price of a certain size hard coal in the city of New York for several years was:

1913 — \$6.50	1917 — \$ 7.75	1921 — \$13.25	1925 — \$14.00
1915 — \$5.50	1919 — \$10.50	1923 — \$13.50	1927 — \$13.75

8. Obtain a record of the number of pupils present in your class each of the days of a recent week and make a graph of it.

Circle Graphs

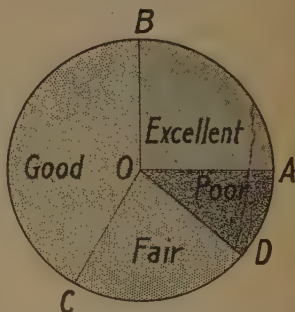
Bar graphs and proportional line graphs are generally used to illustrate any number of *separate* related facts. When, on the other hand, a *completed whole* is given as separated into several parts, the circle may be used to greater advantage.

EXAMPLE. A teacher rated the 36 students in the algebra class as follows: 9 excellent, 15 good, 8 fair, and 4 poor. Draw a circle graph to show these facts.

EXPLANATION. $\frac{9}{36}$, or $\frac{1}{4}$, of the class were rated excellent; $\frac{15}{36}$, or $\frac{5}{12}$, of the class were rated good; $\frac{8}{36}$, or $\frac{2}{9}$, of the class were rated fair; and $\frac{4}{36}$, or $\frac{1}{9}$, of the class were rated poor.

Draw a circle of a convenient size and a radius, as OA , for a base line. Since there are 360° about a point, find $\frac{1}{4}$ of 360° , or 90° . This will represent the number of pupils that were excellent. With the aid of a protractor, draw a radius OB making an angle of 90° with the base line OA . The sector OBA represents the 9 students who were rated excellent.

In the same way lay off successive angles of 150° , 80° , and 40° , and draw the corresponding radii to form the circular sectors that represent the number of students who were good, fair, and poor. If the work is done correctly, the last radius will coincide with OA .



RELATIVE RATINGS OF THE
STUDENTS IN AN ALGEBRA
CLASS

The facts represented in a circle graph must all belong to a single topic of discussion and all the facts must be known.

EXERCISES

Represent the following relations by circle graphs.

1. Frank earned \$24 a week and spent it as follows: \$12 for board and lodging, \$6 saved for future needs, \$1 for insurance, and \$5 for incidentals.

2. The 2400 students registered in a certain school were distributed as follows:

First year,	660	Third year,	590
Second year,	630	Fourth year,	520

3. Mr. Raynor saved \$20,000 and invested it in the following ways: \$1500 in a savings bank, \$8500 in his home, \$4000 in real estate mortgages, \$3500 in bonds, and \$2500 in stocks.

4. James had a library of 80 books, classified as follows: 25 school textbooks, 5 books on history, 8 books on biography, 12 books on travel, and 30 books of fiction.

5. Louise attended school 180 days during the year. She was sick 10 days, worked 60 days, and spent the remainder of the year in rest and recreation.

6. One year the city of New York divided the money it received from taxes in the following proportions: education, 21%; protection of life and property, 16%; health and recreation, 11%; charitable purposes, 5%; public debt, 26%; and all other ways, 21%.

7. One year the population of the United States was divided into the following age groups:

Under 5 yr. of age,	10.9%	25 yr. to 44 yr. of age,	29.6%
5 yr. to 14 yr. of age,	20.8%	45 yr. to 64 yr. of age,	16.1%
15 yr. to 24 yr. of age,	17.8%	All others,	4.8%

8. One year a telephone company reported that it divided each dollar of expense as follows:

Salaries and wages,	51%	Taxes,	8%
Supplies, rents, etc.,	23%	Interest and dividends,	18%

9. The Seventh Street Savings Bank had its funds invested in the following securities: U. S. Government bonds, 14%; state and municipal bonds, 12%; railroad mortgage bonds, 14%; first mortgages, 58%; and miscellaneous securities, 2%.

10. One year 74% of the passenger automobiles manufactured were closed cars, 17.4% were phaetons, and 8.6% were roadsters.

Broken-line and Curved-line Graphs

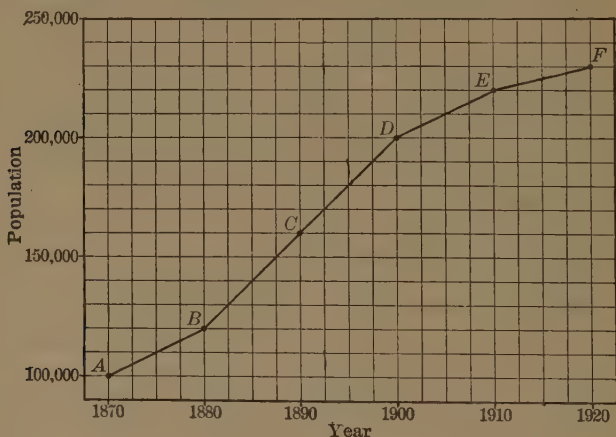
The changes recorded in numerical facts during different seasons of the year or during different years may be shown by broken-line graphs.

EXAMPLE. Represent by a broken-line graph the growth of the population of Louisville, to the nearest 10,000, as follows:

1870 — 100,000	1890 — 160,000	1910 — 220,000
1880 — 120,000	1900 — 200,000	1920 — 230,000

EXPLANATION. Two line scales are drawn at right angles to each other.

The first year mentioned is 1870; hence the first space on the horizontal scale represents 1870. 4 spaces are used to represent each 10 yr.



POPULATION OF THE CITY OF LOUISVILLE FROM 1870 TO 1920

The lowest population given is 100,000, hence this number is used for the first space on the vertical scale. Each vertical space is taken as 10,000.

The point *A* represents the year 1870 and the population 100,000. It is located on the intersection of the two perpendicular lines drawn through the scale points representing these numbers. The other points are determined in a similar way. In constructing a graph, the points *A*, *B*, *C*, etc. should not be labeled, but should simply be indicated by small dots. The graph is completed by drawing lines to connect the successive points.

Directions to be observed in drawing broken-line graphs.

Draw two line scales perpendicular to each other. These perpendicular scales are called **axes**. If one of the facts to be shown by the graph is a time element, it is customary to let the horizontal axis show this.

Choose a convenient scale to represent each of the two elements given in the table of facts. It is not necessary to represent the zero of either scale unless 0 is included in the data given in the table. Read the data carefully to determine the best starting point for the graph.

Locate the successive points of the graph and connect them by straight lines.

A poorly constructed graph is of very little value. Be accurate and neat. Indicate the scale used. State the title of the graph.

EXERCISES

Construct broken-line graphs to represent the following :

1. The number of students enrolled in a country school was :

1918 — 15	1920 — 20	1922 — 21	1924 — 18	1926 — 17
1919 — 18	1921 — 24	1923 — 19	1925 — 21	1927 — 15

2. The average retail price of automobiles for several years is given by the following table :

YEAR . . .	1918	1919	1920	1921	1922	1923	1924	1925	1926
PRICE . . .	\$1150	\$1175	\$1275	\$975	\$875	\$825	\$825	\$875	\$885

3. The prices a farmer received for a bushel of wheat during certain years were :

1916 — \$1.08	1919 — \$2.06	1922 — \$.95	1925 — \$1.16
1917 — \$1.57	1920 — \$2.33	1923 — \$1.04	1926 — \$1.58
1918 — \$2.01	1921 — \$1.49	1924 — \$.96	

4. The consumption of coffee per capita in the United States was :

YEAR	1912	1914	1916	1918	1920	1922	1924	1926
POUNDS PER CAPITA	9.3	10.0	11.0	10.3	13.0	11.6	12.4	12.8

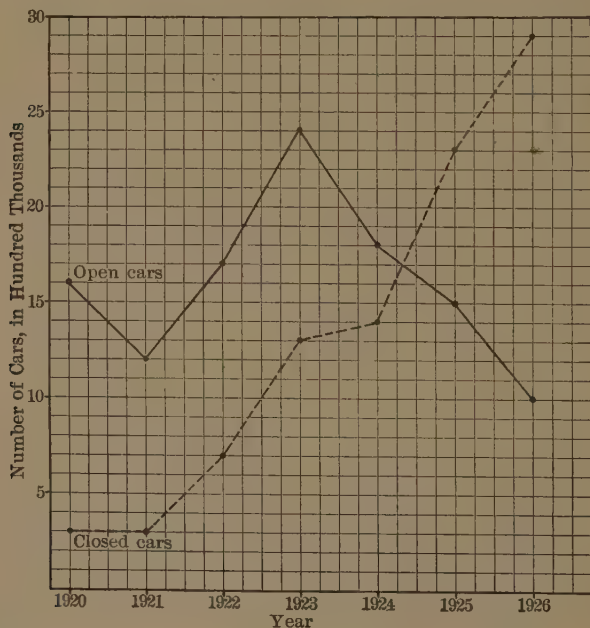
5. The growth of the density of population per square mile in the United States is given as follows :

YEAR	1860	1870	1880	1890	1900	1910	1920
DENSITY	10.6	13.0	16.9	21.2	25.6	30.9	35.5

From the graph read the approximate density in 1905. Extend the graph and determine the probable density in 1930.

6. The number of open and closed passenger automobiles made during certain years, to the nearest 100,000, is given as :

YEAR	1920	1921	1922	1923	1924	1925	1926
OPEN CARS . . .	16	12	17	24	18	15	10
CLOSED CARS . .	3	3	7	13	14	23	29



NUMBER OF PASSENGER AUTOMOBILES MADE DURING THE YEARS
1920 TO 1926

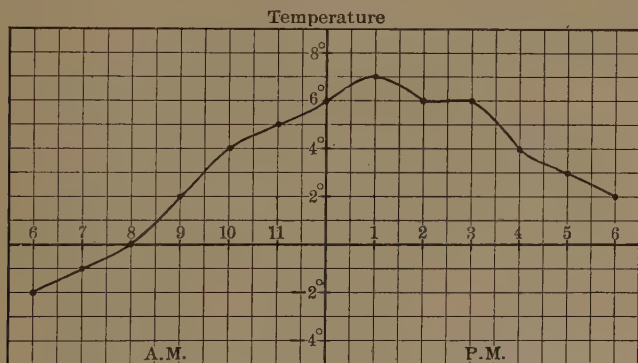
In **curved-line graphs**, succeeding points are connected by lines that form a smooth curve passing through all the points. Curved lines may be used to represent a succession of facts when the change from one fact to the next is gradual or continuous.

Thus, a change in temperature from one hour to the next or a change in weight of a boy from 90 lb. to 98 lb. may be represented by a curved line.

Positive and negative numbers may be represented on a graph, as illustrated in the following example.

EXAMPLE. Show by a graph the changes in temperature for a day in January, as recorded in the following table:

				A.M.			M.				P.M.		
TIME	6	7	8	9	10	11	12	1	2	3	4	5	6
TEMP.	-2°	-1°	0°	2°	4°	5°	6°	7°	6°	6°	4°	3°	2°



CHANGES IN TEMPERATURE ONE DAY IN JANUARY

EXPLANATION. Two axes are drawn at right angles to each other. Since the degrees are positive and negative quantities and the A.M. time may be considered as negative and the P.M. time as positive, the axes are extended below and to the left of the point of intersection.

The point where the axes intersect is called the **origin**.

1. What was the approximate temperature at 10.30 A.M.? at 2.30 P.M.? At what time was the temperature 1°? 3°?

2. If the graph were extended to the right, what temperature would be indicated for 7 P.M.?

EXERCISES

Draw graphs to represent the following statements:

1. The average temperatures recorded for a week in January at Jamesville were as follows:

Sunday, 18° Tuesday, 15° Thursday, -3° Saturday, -2°
Monday, 20° Wednesday, 5° Friday, -10°

2. The monthly mean temperatures recorded at Chicago and Denver one year were as follows:

	JAN.	FEB.	MAR.	APR.	MAY	JUNE
Chicago	25°	27°	36°	48°	58°	68°
Denver	30°	33°	39°	47°	56°	66°
	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
Chicago	74°	73°	66°	55°	41°	30°
Denver	72°	71°	63°	51°	40°	32°

On the same set of axes construct two curved-line graphs to represent these temperatures. Represent Chicago by a solid line and Denver by a dash line.

During what months did Chicago have the higher temperatures? During what months did the cities have the same temperatures?

3. The depth of a river was measured at several points in crossing from one side of the river to the other and the following record was made:

Feet from starting point	. 0	20	40	60	80	100	120
Depth of river in feet	. . 0	8	15	25	23	10	0

HINT. Represent the depths as negative numbers below the horizontal axis.

From the graph estimate the depth of the river 25 ft. from each bank. Where was the river approximately 20 ft. in depth?

4. The average weight of normal girls and boys at different ages is given in the table below. Construct two curved-line graphs on the same set of axes to represent both sets of weight.

Age in Years . . .	6	7	8	9	10	11	12	13	14	15	16
Weight of Girls in Pounds	43	47	52	57	62	69	78	89	98	106	112
Weight of Boys in Pounds	45	49	54	59	65	70	77	85	95	107	121

From the graph determine the ages during which a girl should weigh more than a boy of the same age. At what ages do they weigh the same? Determine the age of a girl who weighs 92 lb. How much should a boy $12\frac{1}{2}$ yr. old weigh?

5. Construct two graphs on the same set of axes to show the comparative number of sailing vessels and steamships built in the United States from 1913 to 1924.

YEAR	1913	1914	1915	1916	1917	1918	1919
Sailing vessels	72	51	51	34	64	115	84
Steamships	1004	778	140	129	192	309	891
YEAR	1920	1921	1922	1923	1924	1925	1926
Sailing vessels	115	70	45	34	16	28	20
Steamships	965	451	131	127	94	74	25

What effect did the World War have upon the number of vessels built from 1913 to 1917? from 1918 to 1921? Explain.

6. The relative retail prices for food in the United States, based upon the price for 1913 as 100, were as follows:

1911 — 92	1915 — 101	1919 — 186	1923 — 146
1912 — 98	1916 — 114	1920 — 203	1924 — 146
1913 — 100	1917 — 146	1921 — 153	1925 — 157
1914 — 102	1918 — 168	1922 — 142	1926 — 154

Statistics

The groups of number facts that you have been representing graphically are called **statistics**. The numbers in any one group are the *data* that have been gathered for consideration.

Thus, the annual report of a certain railroad showed that its revenues from freight business and passenger business for certain years were, to the nearest million dollars, as follows :

YEAR	1917	1918	1919	1920	1921	1922	1923	1924	1925	1926
Passenger .	36	43	53	63	53	49	53	48	44	45
Freight . .	117	131	145	168	160	158	166	168	175	180

Statistical data may be represented either by a *table* of numbers, as shown above, by a *graph* based on the given data, or by a *formula* showing the relations between the numbers. The graph shows at a glance facts not readily seen from the table ; but a study of tables of statistics is useful at times.

Consider the following record made by the 27 students of Miss Simpson's class in an exercise of 20 linear equations :

NAMES	SCORES	NAMES	SCORES
Harry Austin	15	Agnes Johnson	20
Cora Best	13	Mary Kirby	17
Frank Bode	16	Fred Larson	12
Erwin Brown	17	Roy McNally	17
Enid Burt	14	Alice Merritt	16
John Doherty	19	John O'Rourke	17
Will Dunloy	15	Will Peake	16
Will Fricke	18	Sarah Smith	16
John Hanly	16	Fred Steen	18
Stella Harrison	18	Howard Watson	15
Lucy Hart	12	Anna Williams	14
Robert Hebard	15	Jessie Wilson	17
Rose Hunt	19	Chester Worth	16
Walter Janson	11		

The table arranged alphabetically, as on page 122, gives us information concerning the individual members of the class, but does not give us much information concerning the class as a whole. The standing of the class may be seen better if the names are arranged in *order of rank*:

NAMES	SCORES	NAMES	SCORES
Agnes Johnson . . .	20	Will Peake	16
John Doherty	19	Sarah Smith	16
Rose Hunt	19	Chester Worth	16
Will Fricke	18	Harry Austin	15
Stella Harrison . . .	18	Will Dunloy	15
Fred Steen	18	Robert Hebard	15
Erwin Brown	17	Howard Watson	15
Mary Kirby	17	Enid Burt	14
Roy McNally	17	Anna Williams	14
John O'Rourke	17	Cora Best	13
Jessie Wilson	17	Fred Larson	12
Frank Bode	16	Lucy Hart	12
John Hanly	16	Walter Janson	11
Alice Merritt	16		

A glance at the table above shows the ranking order of each pupil. It shows also the number of students that received equal scores.

For the sake of comparison with the class, the **average number** of problems worked by the pupils may be found by dividing the total of the scores by the number of pupils in the class. For Miss Simpson's class the average is found to be 15.9, or, in round numbers, 16.

The average of a class represents the **central tendency**. That is, it is a number about which the others may be grouped for comparison. The average, however, does not always present a fair number for comparison, especially if the extreme data differ much from the rest of the data. This may be seen in the following case.

An Easter collection was taken to pay a deficit in the running expenses of a church, with the result shown below :

NUMBER OF PERSONS	AMOUNT EACH GAVE	NUMBER OF PERSONS	AMOUNT EACH GAVE
50	\$.50	5	\$ 5.00
10	.75	2	10.00
10	1.00	1	150.00

The average contribution was slightly less than \$3.05 ; but this does not correctly represent what most of the people gave, as the result is unduly influenced by the one who gave a much larger sum of money than the rest. In this instance \$.50 is a more representative sum of money, since 50 persons contributed that amount and only 28 persons gave more.

Consider the table on page 123 that gives the order of rank of each pupil in Miss Simpson's class. The pupil who is in the middle of the class, when arranged according to rank, is called the **median** of the class. As there are 27 pupils in the class, the 14th pupil, or Alice Merritt, represents the median. She worked 16 equations correctly.

The middle number in a series of numbers arranged according to rank is called the **median number**. If there is no middle member in the series, then a number halfway between the two middle numbers is called the median. The median may be used also to represent the central tendency.

EXERCISES

1. In ex. 1, page 117, determine the average number of students that attended the school during the years mentioned. What was the median number?

2. In ex. 4, page 117, find the average number of pounds of coffee consumed per person during the years mentioned. What was the median number?

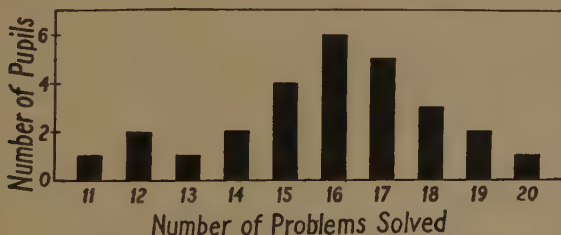
The central tendency might be indicated also by finding the number that occurs the most frequently in a series. To find the number of pupils who received the same score, the data given on page 123 may be tabulated as below:

NUMBER OF PUPILS	SCORE	NUMBER OF PUPILS	SCORE
1	20	4	15
2	19	2	14
3	18	1	13
5	17	2	12
6	16	1	11

Such a table is called a **frequency table** as it indicates the frequency with which a certain number occurs in a series.

The number 16 occurs the greatest number of times in the table above. Such a number is called the **mode** or **modal number**, as it is *according to the mode or fashion*.

The mode may be represented by a bar graph, as shown below:



ACHIEVEMENT OF MISS SIMPSON'S ALGEBRA CLASS

You have seen that the central tendency of a group of numbers may be represented in three ways: by the *average* of the group, by the *median*, or by the *mode*. In the case of the algebra class that you have considered, the same number represents each central tendency.

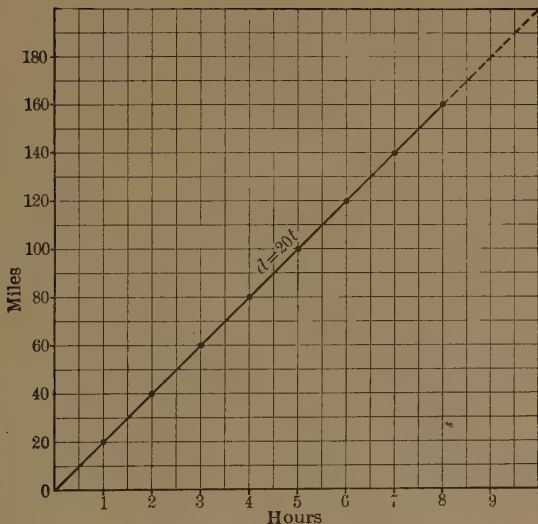
EXERCISES

1. Tell what is meant by the terms: (a) ranking order table; (b) frequency table; (c) central tendency.
2. What is the median amount that each person contributed to the Easter collection mentioned on page 124? What is the modal amount?
3. In ex. 1, page 113, what was the average number of boys that reported for each sport? the median number? the modal number?
4. In ex. 6, page 121, during which year was the price for food nearest the average price for the years mentioned? Which year represents the median price? the modal price? Which of these prices best represents the central tendency?
5. Make a frequency table of the marks that your algebra class received for the last written lesson.
6. From the table of ex. 5, determine the average score of the class; the median score; the modal score.
7. Construct bar graphs to represent the achievement of your algebra class.
8. Obtain a record of the weight, to the nearest pound, of each boy and of each girl in your algebra class. From the data make two frequency tables of the weights of the pupils in your class.
9. Construct bar graphs to show the facts of the two frequency tables of ex. 8.
10. From the tables in ex. 8, determine whether the median weight or the average weight best represents the weight of a boy or of a girl in your class. In what way do the bar graphs show this?
11. State the advantages and the disadvantages of each of the three methods of determining the central tendency of a group of numbers.

Graphs of Formulas

The formula $d = 20t$ shows the distance an automobile moving at the rate of 20 mi. an hour will travel in t hours. If you substitute the values 1, 2, 3, etc. for t , you obtain a corresponding series of values for d , thus:

$t =$	0	1	2	3	4	5	6	7
$d =$	0	20	40	60	80	100	120	140



GRAPH OF THE FORMULA $d = 20t$

The method for constructing the graph above is similar to that for broken-line graphs, explained on page 116.

Direct variation is shown by graphs constructed in this way. A study of the graph above indicates that the distance varies directly as the time. That is, if the number of hours is doubled, the distance passed over is doubled. (See page 105.)

Graphs of formulas of the first degree are **straight lines**.

EXERCISES

Construct graphs to represent the relations shown by the following formulas :

1. The formula $c = 25n$ indicates the cost of n yards of ribbon at 25¢ a yard. Plot the graph of c from $n = 0$ to $n = 15$.

2. (a) The interest formula $i = 100 \times \frac{5}{100} \times t$, or $i = 5t$, indicates the interest received on \$100 at 5% for t years. Plot the graph of i from $t = 0$ to $t = 15$.

(b) From the graph determine the interest on \$100 at 5% for 3 yr.; for $7\frac{1}{2}$ yr.

3. The formula $C = 2\pi r$ shows the relation between the circumference of a circle and the radius. Use $\pi = 3\frac{1}{7}$. Plot the graph of C from $r = 0$ to $r = 10$.

4. (a) The formula for the area of a parallelogram having an altitude of 6 in. and a base of b inches is $A = 6b$. Plot the graph of A from $b = 0$ to $b = 15$.

(b) From the graph determine the area when $b = 2$; when $b = 6\frac{1}{2}$; when $b = 11\frac{1}{2}$.

5. Water weighs about 62.5 lb. per cubic foot. The formula $W = 62.5V$ states this relation if the volume V is given in cubic feet. Plot the graph of W from $V = 0$ to $V = 10$.

6. (a) The formula $c = 45 + 3(n - 10)$ indicates the cost of sending a telegram of n words if the cost is 45¢ for the first 10 words and 3¢ for each additional word. Plot the graph for c from $n = 10$ to $n = 35$.

(b) From the graph determine the cost of sending a message of 25 words; of 18 words. How many words may be sent for 87¢? Explain the cost of sending 7 words.

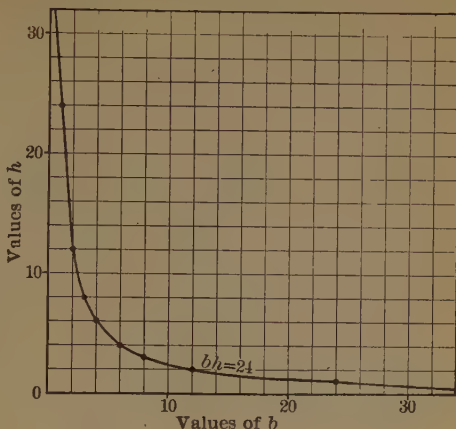
7. The formula $C = \frac{5}{9}(F - 32)$ expresses the relation existing between temperature readings on a Fahrenheit thermometer and the corresponding readings on a centigrade thermometer. Plot the graph of C from $F = 0^\circ$ to $F = 100^\circ$.

8. (a) The formula $S = C + \frac{30}{100}S$ states that the selling price, S , is equal to the cost, C , plus a profit of 30% of the selling price.

Plot the graph of S from $C = 0$ to $C = 10$.

(b) From the graph determine the cost of an article that sells for \$2.50.

9. The graph of the area formula, $A = bh$, is given at the right for $A = 24$ square units. From the graph determine the value of b when $h = 1$; when $h = 2$; when $h = 3$; when $h = 6$.



GRAPH OF THE FORMULA $bh = 24$

As the value of h increases, the value of b decreases. If h is made 3 times as large, b becomes only $\frac{1}{3}$ as large.

This graph and others similar to it represent *inverse variation*.

10. Construct the graph of the formula for uniform motion, $d = Vt$, when $d = 48$ ft. and t changes from 1 min. to 48 min.

11. Show that the graph constructed in ex. 10 may be used to represent the cost formula $c = np$ for $c = \$48$.

12. (a) The formula $d = 16t^2$ gives the distance in feet a body will fall in t seconds. Construct the graph of this formula for values of t from 0 to 4.

(b) From the graph determine the value of d when $t = 1\frac{1}{2}$; when $t = 3$.

This graph shows that d changes as the square of t ; that is, if the value of t is doubled, the value of d is made 4 times as great.

13. Construct a graph of the formula $A = \pi r^2$ for values of r from 0 to 4, using $\pi = 3.1$.

Graphs of Equations

The formulas that you studied on the two preceding pages are special forms of equations. Such formulas, in general, are concerned with positive numbers only; but equations involve negative numbers also, as is shown in the discussion below.

You have studied equations of the first degree that involve one unknown number. Many equations involve two or more unknown numbers. For example, the equation $x + 2y = 4$ has two unknown numbers and the equation $x + 2y + 3z = 8$ has three unknown numbers.

The quantities x and y of the equation $x + 2y = 4$ are **variables**. The known term 4 is a constant, called the **constant term** or the **absolute term**.

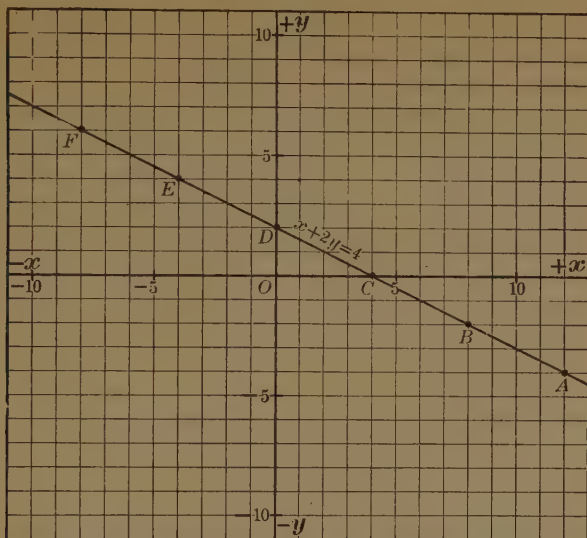
If values for x are substituted in the equation $x + 2y = 4$, corresponding values of y can be obtained and the following table may be constructed.

x	y
12	- 4
8	- 2
4	0
0	2
- 4	4
- 8	6

This table can be extended indefinitely in either direction, for there is no limit to the number of pairs of values of x and y that will satisfy the equation. On the other hand, it is not possible to substitute any value for x and any other value taken at random for y . For example, $x = 5$ and $y = 7$ would not satisfy the equation; for if $x = 5$, then y must equal $-\frac{1}{2}$, and if $y = 7$, then x must equal -10 . The values of x and y must satisfy the equation $x + 2y = 4$.

In the equation $x = 4 - 2y$, x is said to be a function of y . Similarly, in the equation $y = \frac{1}{2}(4 - x)$, y is a function of x . In the equation $x + 2y = 4$, either variable is a function of the other, since if a value is substituted for one variable, the value of the other variable is determined. (See page 103.)

EXAMPLE. Draw the graph of the equation $x + 2y = 4$.



SOLUTION. Solve the equation for x and make a table of values for x and y like the table on page 130.

Each of the pairs of values in the table represents a point (x, y) of the graph, designated as the points $(12, -4)$, $(8, -2)$, etc. It is customary always to state first the value of x , then the value of y .

Draw two perpendicular axes on squared paper. The vertical axis is the y -axis and the horizontal axis is the x -axis. The point where the two axes cross is the origin.

All positive values of x are plotted to the right of the y -axis, a zero value of x is on the y -axis, and negative values of x are plotted to the left of the y -axis. Similarly, positive values of y are plotted above the x -axis, a zero value of y is on the x -axis, and negative values of y are below the x -axis.

Locate point $A = (12, -4)$ by finding the point on the x -axis corresponding to $x = 12$ and then locating point A four spaces below the x -axis, since $y = -4$. In a similar way locate points B , C , D , E , and F . Connect the points by a line.

Since the equation $x + 2y = 4$ is of the first degree in two variables, the points will all lie on the same straight line. For this reason equations of the first degree in two variables are called **linear equations**.

The two numbers that locate a point are called the **coördinates** of the point. The x distance is sometimes called the **abscissa** of the point and the y distance, the **ordinate** of the point. Similarly, the axes are sometimes referred to as the **coördinate axes**, the x -axis being the axis of abscissas and the y -axis, the axis of ordinates.

EXERCISES

1. Locate each of the following points:

(a) $x = 2, y = 3$

(d) $x = -2, y = -7$

(b) $x = -3, y = 5$

(e) $x = 0, y = 5$

(c) $x = 5, y = -8$

(f) $x = 4, y = 0$

2. Locate each of the following points:

$A = (-1, 0)$ $B = (0, 1)$ $C = (1, 2)$ $D = (2, 3)$ $E = (3, 4)$

3. Notice that each value of y in ex. 2 is 1 more than the corresponding value of x . State this relation in the form of an equation.

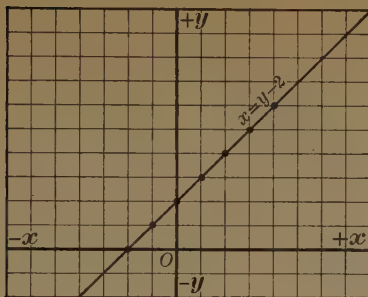
4. Construct the graph of $y = x + 1$. Note that the y -value of each point of the graph is 1 more than its corresponding x -value.

Locus of a point.

EXAMPLE. Graph the equation $x = y - 2$.

The equation $x = y - 2$ is the algebraic statement that the value of x is always 2 less than that of y . This equation is the algebraic equivalent of the indefinite table of values:

$x =$	- 2	- 1	0	1	2	3	4	etc.
$y =$	0	1	2	3	4	5	6	etc.



The graph of the equation $x = y - 2$ is the straight line that contains every point x that is 2 less than the corresponding value of y . The graph is the place where all these points lie; it is their location and is called the *locus* of all such points.

The **locus of a point** is the line or group of lines that contains all the points that answer a certain condition and no points that do not answer this condition. In the above instance, the condition is that all the x -points shall be 2 less than the corresponding y -points.

EXERCISES

Construct the graphs for the following equations:

- | | | |
|-----------------|-----------------|-------------------|
| 1. $y = 3x - 2$ | 4. $y = 2x + 3$ | 7. $2x - 3y = 2$ |
| 2. $2x = 3y$ | 5. $x = 2y - 5$ | 8. $5x + y = 10$ |
| 3. $x = 3y + 1$ | 6. $3x = y + 3$ | 9. $3x = 2y - 12$ |

10. Write the equation that states that the x distance of a point is always equal to the y distance. Construct the locus of these points.

11. State ex. 1 to 5 as locus problems.

12. What is the value of x at the point where the graph of $y = 3x - 6$ crosses the x -axis? What is the value of y at the point where the graph crosses the y -axis?

Graphic Solution of Systems of Equations

As you have seen, an equation having two variables is satisfied by an unlimited number of sets of values of the two variables involved. Such an equation is called **indeterminate**.

The equation $x + 2y = 8$ is satisfied by the sets of values:

$x =$	10	8	6	4	2	0	etc.
$y =$	-1	0	1	2	3	4	etc.

The equation $3x - y = 3$ is satisfied by the sets of values:

$x =$	-2	-1	0	1	2	3	etc.
$y =$	-9	-6	-3	0	3	6	etc.

Each of the equations $x + 2y = 8$ and $3x - y = 3$ is satisfied by an unlimited number of other sets of values. An examination of the two tables of values shows that each equation is satisfied by the same pair of values ($x = 2, y = 3$).

Two equations containing two variables that are satisfied by the same values of the variables involved form a **system of simultaneous equations** or simply a **system**, or **set**, of equations.

The pairs of values that satisfy each of the equations in the system is called the **solution of the system**.

It is often tedious to make tables of values to find pairs of values that satisfy each of the equations of a *system of equations*. The work may be shortened by finding the coördinates of the points of intersection of the graphs of the two equations.

Since two points determine a straight line, it is necessary to plot only two points of a linear equation in order that the equation may be located. These two points should be several spaces apart. Frequently it is convenient to find the two points corresponding to the values $x = 0$ and $y = 0$. It is a good check of the work, however, to locate a third point to see that it lies on the line determined by the first two points.

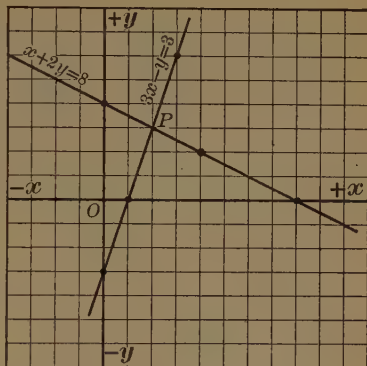
EXAMPLE. Solve graphically the system of equations:

$$\begin{cases} x + 2y = 8 \\ 3x - y = 3 \end{cases}$$

SOLUTION. To graph the equation $x + 2y = 8$, plot the points (0, 4) and (8, 0). Check the graph by plotting the point (4, 2).

To graph the equation $3x - y = 3$, plot the points (0, -3) and (1, 0). Check the graph by plotting the point (3, 6).

The graphs of the two equations cross only at the point (2, 3). That is, the locus of each equation passes through this point. Hence the point ($x = 2$, $y = 3$) is the graphic solution of the system of equations.



Check the result by substituting the values of x and y in each of the original equations:

$$2 + 2(3) = 8, \quad \text{or } 8 = 8$$

$$3(2) - 3 = 6 - 3, \quad \text{or } 3 = 3$$

EXERCISES

Solve by graphic methods each of the following systems of equations:

1. $\begin{cases} x + 2 = y \\ x + y = 6 \end{cases}$

2. $\begin{cases} x - y = 1 \\ x + y = 9 \end{cases}$

3. $\begin{cases} x + y = 3 \\ x + 2y = 4 \end{cases}$

4. $\begin{cases} 4x - y = 0 \\ x - y = 3 \end{cases}$

5. $\begin{cases} x + y = -3 \\ x + 12 = 2y \end{cases}$

6. $\begin{cases} 2x - 5y = 0 \\ 2x + y = 12 \end{cases}$

7. $\begin{cases} 3x - 2y = 7 \\ y = x - 2 \end{cases}$

8. $\begin{cases} 4x - y = 7 \\ 2x + y = 8 \end{cases}$

9. $\begin{cases} x = 2y \\ x + 2y = 6 \end{cases}$

10. $\begin{cases} 2x + 4y = 19 \\ 4x - 2y = 3 \end{cases}$

11. $\begin{cases} x + 2y = 0 \\ 3x - 2y = 20 \end{cases}$

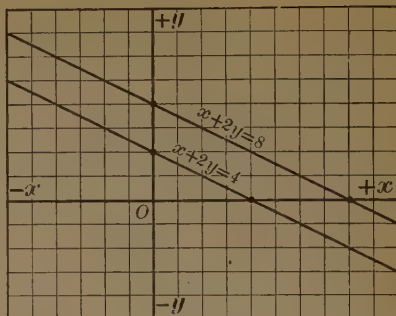
12. $\begin{cases} 2x - 5y = 3 \\ 4x = 7y \end{cases}$

Special sets of equations.

EXAMPLE 1. Construct the graph of the equations:

$$\begin{cases} x + 2y = 8 \\ x + 2y = 4 \end{cases}$$

The graph of the two equations is shown at the right. The two lines are parallel and will not intersect; that is, they have no points in common. This is what might be expected from a study of the two equations, since it is not possible for the sum of one number, x , and twice another number, y , to be 8 and also 4 at the same time.

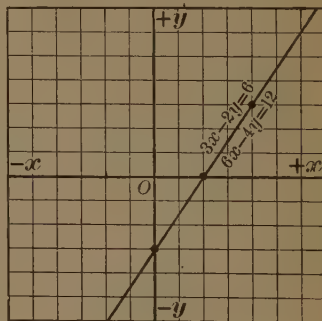


Such equations as $x + 2y = 8$ and $x + 2y = 4$ are called **inconsistent equations**. They have no solution in common.

EXAMPLE 2. Construct the graph of the equations:

$$\begin{cases} 3x - 2y = 6 \\ 6x - 4y = 12 \end{cases}$$

The graph of the two equations at the right consists of one line. A study of the equations shows that the second equation is obtained by multiplying each of the terms of the first equation by 2. Hence every set of values that satisfies $3x - 2y = 6$ also satisfies $6x - 4y = 12$.

**EXERCISES**

Construct the graphs of the following equations. If the equations are simultaneous, determine their solution:

1. $\begin{cases} 2x - y = 6 \\ 2x - 10 = y \end{cases}$
2. $\begin{cases} 3y + 9 = x \\ 2x - 4 = 6y \end{cases}$
3. $\begin{cases} 3x + y = 3 \\ x + y = 5 \end{cases}$
4. $\begin{cases} 2x + 3y = 6 \\ 6y + 4x = 12 \end{cases}$
5. $\begin{cases} 3x + y = 2 \\ 3x - 2y = 2 \end{cases}$
6. $\begin{cases} x - 3y = 9 \\ 6y + 18 = 2x \end{cases}$

REVIEW EXERCISES

Construct appropriate graphs to represent each of the following sets of facts:

1. Miss Halstead earned \$1800 a year and divided it as follows: board and lodging, \$900; clothes, \$350; investments, \$150; vacation and recreation, \$350; incidentals, \$50.

2. The average price of eggs per dozen that the farmer received for certain years is given as:

Year . .	1917	1918	1919	1920	1921	1922	1923	1924	1925	1926
Price . .	37¢	48¢	53¢	61¢	55¢	38¢	39¢	35¢	49¢	36¢

3. Mary spends her day approximately as follows: $1\frac{1}{2}$ hr. at meals, 6 hr. at school, $2\frac{1}{2}$ hr. studying, 5 hr. in recreation, and 9 hr. in sleep.

4. The following table shows the earnings and expenditures of a boy from April to October:

Month . . .	APRIL	MAY	JUNE	JULY	AUG.	SEPT.	OCT.
Earnings . .	\$10	\$12.50	\$8	\$35	\$20	\$8.50	\$12
Expenditures	\$12.50	\$8	\$5.50	\$15.20	\$25	\$9.75	\$6.75

On the same diagram make two graphs, one of the earnings and the other of the expenditures. (Use a differently colored line or a dotted line to represent the expenditures.)

5. The lengths of some of the principal canals in the United States are:

Erie	340 mi.	Illinois and Michigan . . .	95 mi.
Cayuga and Seneca . . .	92 mi.	Chicago Drainage . . .	39 mi.
Chesapeake and Ohio . .	185 mi.	Schuyllkill	50 mi.

6. State three ways of presenting statistics. Give illustrations of each.

7. When speaking of statistics, what is meant by central tendency? Illustrate this term by means of ex. 2 above.



Fun with Figures



1. You were half as old as I am when I was as old as you are now. When you are as old as I am now, the sum of our ages will be 54 yr. How old are we?

2. If a fourth of 20 were 3, what would a third of 50 be?

3. What figure placed anywhere in, before, or after the number 315,025 will make the new number divisible by 9?

4. What similarity do you notice in the products of 142,857 by each of the numbers 2, 3, 4, 5, and 6. Find the sum of all the products. Multiply the sum by 7.

5. A man borrowed as much money as he had and then spent \$10. He again borrowed as much as he then had and then spent \$10. He repeated this a third time and had nothing left. How much money did he have at first?

6. A man has \$20,000. If he spends $\frac{1}{2}$ of this to-day and $\frac{1}{2}$ of the remainder each day after to-day, how many days will his money last?

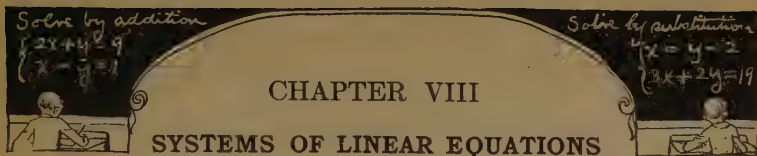
7. Three girls went to market with apples for sale. Ruth sold 50 apples, Clara sold 30 apples, and Alice sold 10 apples. They sold the apples at the same rate and each received the same amount of money. At what rate did they sell their apples?

HINT. They sold multiples of 7 apples at one rate and the remainder at so much for each apple. Each girl received 30¢.

8. John Corfield deposited \$100 in a bank. He later withdrew:

First	\$ 35, leaving a balance of \$ 65
Second	\$ 30, leaving a balance of \$ 35
Third	\$ 25, leaving a balance of \$ 10
Fourth	\$ 10, leaving a balance of <u>0</u>
Total	\$100 \$110

Explain the extra \$10.



You have studied the algebraic solution of linear equations in one unknown number and the graphic solution of linear equations involving two unknown numbers.

The graphic solution of systems of equations is often tedious; hence it is well to be familiar with the algebraic method of solving such equations.

Two equations involving two unknown numbers that are satisfied by the same set of values of the unknown numbers form a **system of simultaneous equations**. (See page 134.) Thus, the equations $\begin{cases} 2x + y = 8 \\ 3x - y = 7 \end{cases}$ form a system of simultaneous equations. If you draw the graph of this set of equations, you will find that the two lines cross at the point (3, 2). Hence the equations are satisfied simultaneously by the same set of values.

A single equation involving two unknown numbers is satisfied by an unlimited number of sets of values of the letters involved. Thus, the equation $2x + y = 8$ is satisfied by an infinite number of sets of values of x and y , among which are the following:

$x =$	5	4	3	2	1	0	- 1	- 2	etc.
$y =$	- 2	0	2	4	6	8	10	12	etc.

Such an equation is called **indeterminate** because *any* of the sets of values will satisfy the equation, but no one set of values is determined.

Again, the equation $3x - y = 7$ is satisfied by the values:

$x =$	- 3	- 2	- 1	0	1	2	3	4	etc.
$y =$	- 16	- 13	- 10	- 7	- 4	- 1	2	5	etc.

The set ($x = 3, y = 2$) is found in the table above as well as in the table on page 139; hence it satisfies both equations at the same time (simultaneously) and is called the *solution* of the pair of equations. The graph of such a system of equations shows that it has only one pair of values that will satisfy both equations. (See page 135.)

Algebraic Solution of Systems of Linear Equations

EXAMPLE. Solve by algebraic methods the system of equations:

$$\begin{cases} 2x + y = 8 \\ 3x - y = 7 \end{cases}$$

This system might be solved by selecting a common pair of values from a table of values for x and y . This, however, may be a slow and difficult process. The easiest method of solution is given below.

SOLUTION. Find from the two equations one equation having only one unknown number. This may be done by adding the two equations.

$$\begin{cases} 2x + y = 8 & (1) \\ 3x - y = 7 & (2) \end{cases}$$

$$(1) + (2): \quad \quad \quad \begin{array}{r} 2x + y = 8 \\ 3x - y = 7 \\ \hline 5x = 15 \end{array} \quad (3)$$

$$x = 3 \quad (4)$$

$$\text{Substitute 3 for } x \text{ in (1):} \quad 2(3) + y = 8 \quad (5)$$

$$\text{Solve (5):} \quad \quad \quad y = 2 \quad (6)$$

$$\text{Hence the solution is } \begin{cases} x = 3 \\ y = 2 \end{cases}$$

CHECK. Check the result by substituting the values for x and y in (1) and (2).

$$\text{Substitute } x = 3, y = 2 \text{ in (1): } 2(3) + 2 = 6 + 2, \text{ or } 8 = 8$$

$$\text{Substitute } x = 3, y = 2 \text{ in (2): } 3(3) - 2 = 9 - 2, \text{ or } 7 = 7$$

Note that y disappears in equation (3) above. That is, adding (1) and (2) *eliminates* y from the equations.

Elimination by addition or subtraction. The process by which one equation with one unknown quantity is derived from two equations with two unknown quantities is called **elimination**.

EXAMPLE. Solve the system:
$$\begin{cases} 3x - 2y = 7 \\ 2x + 5y = 30 \end{cases}$$

SOLUTION. Since the coefficients of x and y are different in the two equations, neither x nor y will be eliminated if the equations are added or if one equation is subtracted from the other. It is necessary first to make the coefficients of one of the variables the same in both equations. If the first equation is multiplied by 2 and the second equation is multiplied by 3, the coefficients of x will each be 6, which is the lowest common multiple of 3 and 2.

$$\begin{array}{rcl} & \begin{cases} 3x - 2y = 7 \\ 2x + 5y = 30 \end{cases} & \begin{matrix} (1) \\ (2) \end{matrix} \\ 2 \times (1): & 6x - 4y = 14 & (3) \\ 3 \times (2): & 6x + 15y = 90 & (4) \\ (3) - (4): & \underline{-19y = -76} & (5) \\ & y = 4 & (6) \\ \text{Substitute 4 for } y \text{ in (1):} & 3x - 2(4) = 7 & (7) \\ \text{Simplify (7):} & 3x = 15 & (8) \\ & x = 5 & (9) \end{array}$$

Hence the solution is $\begin{cases} x = 5 \\ y = 4 \end{cases}$

CHECK. (1) $3(5) - 2(4) = 15 - 8$, or $7 = 7$
 (2) $2(5) + 5(4) = 10 + 20$, or $30 = 30$

NOTE. In equations (3) and (4), the coefficients of x are each 6. Equation (5) contains only the one unknown number y and the known term -76 . It is obtained by subtracting (4) from (3).

In a similar way, y may be eliminated from (1) and (2) by multiplying (1) by 5 and (2) by 2 and then adding the resulting equations.

ORAL EXERCISES

1. When may one of the unknown quantities be eliminated by addition? by subtraction?

2. In the example on page 140 how could you eliminate the x 's?

3. Mention the various steps used in elimination by addition or subtraction.

EXERCISES

Solve the following systems of equations and check the roots.
Ex. 1 to 12 are the same as those solved by graphic methods on page 135.

$$1. \begin{cases} x + 2 = y \\ x + y = 6 \end{cases}$$

$$2. \begin{cases} x - y = 1 \\ x + y = 9 \end{cases}$$

$$3. \begin{cases} x + y = 3 \\ x + 2y = 4 \end{cases}$$

$$4. \begin{cases} 4x - y = 0 \\ x - y = 3 \end{cases}$$

$$5. \begin{cases} x + y = -3 \\ x + 12 = 2y \end{cases}$$

$$6. \begin{cases} 2x - 5y = 0 \\ 2x + y = 12 \end{cases}$$

$$7. \begin{cases} 3x - 2y = 7 \\ y = x - 2 \end{cases}$$

HINT. $-x + y = -2$.

$$8. \begin{cases} 4x - y = 7 \\ 2x + y = 8 \end{cases}$$

$$9. \begin{cases} x = 2y \\ x + 2y = 6 \end{cases}$$

$$10. \begin{cases} 2x + 4y = 19 \\ 4x - 2y = 3 \end{cases}$$

$$11. \begin{cases} x + 2y = 0 \\ 3x - 2y = 20 \end{cases}$$

$$12. \begin{cases} 2x - 5y = 3 \\ 4x = 7y \end{cases}$$

$$13. \begin{cases} 5x + 5y = 7 \\ 3x + 7y = 23 \end{cases}$$

$$14. \begin{cases} 2x + y = 3 \\ x - 3y = 5 \end{cases}$$

$$15. \begin{cases} 2x + 3y = 29 \\ 5x = 2y + 6 \end{cases}$$

$$16. \begin{cases} 4x + y = 4 \\ 2x + 3y = -11 \end{cases}$$

$$17. \begin{cases} 5x + 7y = -1 \\ 2x - 3y = -12 \end{cases}$$

$$18. \begin{cases} 4x + 3y = -9 \\ 2x - 5y = 28 \end{cases}$$

$$19. \begin{cases} 2x + 7y = -15 \\ 3x - 2y = -10 \end{cases}$$

$$20. \begin{cases} 3x = 2 - y \\ 5x - 2y = 29 \end{cases}$$

$$21. \begin{cases} 19x + 3y = 4 \\ 7x - 2y = 17 \end{cases}$$

$$22. \begin{cases} 8x = 11 + 2y \\ 4x + 6y = 30 \end{cases}$$

$$23. \begin{cases} 10t + 3w = 98 \\ 3t - 2w = 12 \end{cases}$$

$$24. \begin{cases} 8t + 3u = 27 \\ 3t - 2u = 7 \end{cases}$$

$$25. \begin{cases} 3r + 2s = 1 \\ 2r - 5s = -31 \end{cases}$$

$$26. \begin{cases} 2r + 5s + 11 = 0 \\ 5r - 4s = 25.3 \end{cases}$$

Elimination by substitution. One of the unknown numbers in a system of equations may be eliminated by the method of addition and subtraction or it may be eliminated by another method called **substitution**.

The following examples illustrate the method of substitution.

EXAMPLE 1. Solve the system:
$$\begin{cases} 2x - y = 11 & (1) \\ 3x + 2y = 27 & (2) \end{cases}$$

SOLUTION. Solve (1) for y : $y = 2x - 11$ (3)

Substitute $2x - 11$ for y in (2): $3x + 2(2x - 11) = 27$ (4)

Simplify (4) and solve: $x = 7$ (5)

Substitute 7 for x in (3): $y = 3$

Hence the solution is $\begin{cases} x = 7 \\ y = 3 \end{cases}$

CHECK. Check the result by substituting the values $x = 7$ and $y = 3$ in each of the original equations:

$$2(7) - 3 = 14 - 3, \text{ or } 11 = 11$$

$$3(7) + 2(3) = 21 + 6, \text{ or } 27 = 27$$

EXAMPLE 2. Solve the system:
$$\begin{cases} 7x - 3y = 18 & (1) \\ 2x - 5y = 1 & (2) \end{cases}$$

SOLUTION. Solve (2) for x : $2x = 1 + 5y$ (3)

$$x = \frac{1 + 5y}{2} \quad (4)$$

Substitute $\frac{1 + 5y}{2}$ for x in (1): $7\left(\frac{1 + 5y}{2}\right) - 3y = 18$ (5)

Simplify (5) by removing the parenthesis and multiplying both sides by 2:

$$7 + 35y - 6y = 36$$

$$29y = 29$$

$$y = 1$$

Substitute 1 for y in (4): $x = \frac{1 + 5}{2}$

$$x = 3$$

Hence the solution is $\begin{cases} x = 3 \\ y = 1 \end{cases}$

CHECK. Check the solution by substituting the values $x = 3$ and $y = 1$ in each of the original equations:

$$7(3) - 3 = 18, \text{ or } 18 = 18$$

$$2(3) - 5 = 1, \text{ or } 1 = 1$$

EXERCISES

Solve the following systems of equations by the method of substitution. Check the results:

$$1. \begin{cases} x + 3y = 14 \\ 2x - y = 7 \end{cases}$$

$$2. \begin{cases} 3x - 2y = 13 \\ x + 4y = -5 \end{cases}$$

$$3. \begin{cases} 2x - y = 8 \\ 6x + 2y = 34 \end{cases}$$

$$4. \begin{cases} 3x - y = 14 \\ x + 5y = 10 \end{cases}$$

$$5. \begin{cases} 4x - y = 3 \\ x + 3y = 17 \end{cases}$$

$$6. \begin{cases} t + u = 10 \\ 6t - 7u = 34 \end{cases}$$

$$7. \begin{cases} 3t - 4u = 26 \\ t - 8u = 22 \end{cases}$$

$$8. \begin{cases} x + 10y = 9 \\ 10x + 3y = -7 \end{cases}$$

$$9. \begin{cases} x + 9y = 45 \\ y + 16x = 148 \end{cases}$$

$$10. \begin{cases} 8x + 5y = 23 \\ 9x + 4y = 21 \end{cases}$$

$$11. \begin{cases} 3x + 2y = 17 \\ 5x - y = 37 \end{cases}$$

$$12. \begin{cases} 2x - 17y = 28 \\ x - 5y = 7 \end{cases}$$

$$13. \begin{cases} 3x - 5y = 7 \\ x = y + 3 \end{cases}$$

$$14. \begin{cases} m + 1 = 3n \\ 5n + 9 = 3m \end{cases}$$

$$15. \begin{cases} 3m + n = 17 \\ 2m = 3n - 7 \end{cases}$$

$$16. \begin{cases} 3x + 2y = 23 \\ x + y = 8 \end{cases}$$

$$17. \begin{cases} 2x + 3y = 17 \\ 3x + 2y = 18 \end{cases}$$

$$18. \begin{cases} 4x = 26 + 5y \\ x = 2y + 5 \end{cases}$$

$$19. \begin{cases} 3a + 7b = 16 \\ 2a + 5b = 13 \end{cases}$$

$$20. \begin{cases} 2.5p - q = 13 \\ p - 3q = 13 \end{cases}$$

$$21. \begin{cases} 7p - 3.5q = 7 \\ q - 3p = 0 \end{cases}$$

$$22. \begin{cases} 4m + 5n + 33 = 0 \\ n - 3 = 4m \end{cases}$$

$$23. \begin{cases} 3.2r - 2.4s = 28 \\ 5r + 3s = 10 \end{cases}$$

$$24. \begin{cases} 15r - 7s = 59 \\ 7r + s = 19 \end{cases}$$

EXERCISES

Solve the following equations either by the addition and subtraction method or by the substitution method. Check.

$$1. \begin{cases} 3x + y = 11 \\ 5x = y + 13 \end{cases}$$

$$2. \begin{cases} x = 3 - y \\ x + 2y = 6 \end{cases}$$

$$3. \begin{cases} 3x + y = 10 \\ x + 3y = 6 \end{cases}$$

$$4. \begin{cases} 4x + 5y = -2 \\ 5x + 4y = 2 \end{cases}$$

$$5. \begin{cases} 3x - 2y = 10 \\ 2x + 3y = 11 \end{cases}$$

$$6. \begin{cases} 3x + 2y + 31 = 0 \\ 7x - 3y + 34 = 0 \end{cases}$$

$$7. \begin{cases} 2(x + y) + 3(y - x) = 30 \\ 9x + 2(x + y) = 126 \end{cases}$$

$$8. \begin{cases} 2(3x + 1) - 3(4y + 1) = -3 \\ 3(2x + 1) - 2(4y - 1) = 3 \end{cases}$$

$$9. \begin{cases} 5(x + 2) - 3(y + 1) = 23 \\ 3(x - 2) + 5(y - 1) = 19 \end{cases}$$

$$10. \begin{cases} 5(3x + 2) - 4(6y - 1) = 20 \\ 2(5x - 3) + 7(9y + 1) = 84 \end{cases}$$

$$11. \begin{cases} 2(x + 5) + 3(y + 3) = 42 \\ 3(2x - 3) + 30 = 5(y + 4) \end{cases}$$

$$12. \begin{cases} 4(x + y) - 3(x - y) = 132 \\ 3(x + y) - 2(x - y) = 48 \end{cases}$$

$$13. \begin{cases} .3x + .2y = 9.5 \\ .2x + .3y = 10.5 \end{cases}$$

HINT. Multiply each side of each equation by 10.

$$14. \begin{cases} .2x + 7y = 37 \\ 25x - 48y = 10 \end{cases}$$

$$15. \begin{cases} .5x = y + 2.8 \\ .3x + 5y = 2.8 \end{cases}$$

$$16. \begin{cases} .3x - .2y = .7 \\ .5x + 1.5y = 8.5 \end{cases}$$

$$17. \begin{cases} 3x + 8 = 4y + 2 \\ \frac{4x}{3} + \frac{y}{9} = 3 \end{cases}$$

$$18. \begin{cases} 2.3x + 4.7y = 70 \\ 3.4x + 5.6y = 90 \end{cases}$$

$$19. \begin{cases} 1.6x - 2.05y = .39 \\ 5.2x + 4.1y = 3.42 \end{cases}$$

$$20. \begin{cases} .8x + y = 6 \\ .5x + 5y = 12.5 \end{cases}$$

$$21. \begin{cases} \frac{y}{8} + \frac{3x - 2y}{5} = 25 \\ 2x - 48 = y + 32 \end{cases}$$

Systems of Literal Equations

EXAMPLE. Solve the system: $\begin{cases} ax - by = c & (1) \\ 4ax - 3by = a + 3c & (2) \end{cases}$

SOLUTION. $3 \times (1):$ $3ax - 3by = 3c$ (3)
 $(2) - (3):$ $ax = a$ (4)
 $(4) \div a:$ $x = 1$
 $4 \times (1):$ $4ax - 3by = a + 3c$ (2)
 $(2) - (5):$ $4ax - 4by = 4c$ (5)
 $(2) - (5):$ $by = a - c$ (6)
 $(6) \div b:$ $y = \frac{a - c}{b}$

Therefore the roots are $\begin{cases} x = 1 \\ y = \frac{a - c}{b} \end{cases}$

EXERCISES

Solve the following systems of equations:

1. $\begin{cases} x + y = 5a \\ x - y = 3a \end{cases}$

8. $\begin{cases} 3x - 2y = 4a \\ x = 2y \end{cases}$

2. $\begin{cases} x + y = a \\ x - y = b \end{cases}$

9. $\begin{cases} 2x + 3y = 5a + b \\ 3x + 2y = 5a - b \end{cases}$

3. $\begin{cases} 2x + 3y = a \\ 3x + 2y = b \end{cases}$

10. $\begin{cases} x + 2y = a \\ 2x = y + b \end{cases}$

4. $\begin{cases} 2x + y = a + 1 \\ x - 2y = a - 1 \end{cases}$

11. $\begin{cases} ax + y = m \\ y + n = ax \end{cases}$

5. $\begin{cases} x + 2y = a \\ 2x - y = b \end{cases}$

12. $\begin{cases} ax + 3by = 8 \\ 4 - 3ax = by \end{cases}$

6. $\begin{cases} x = a - 2y \\ x + 1 = b + 2y \end{cases}$

13. $\begin{cases} px = 3 + 2qy \\ 2px = 11 - qy \end{cases}$

7. $\begin{cases} 3x + 2y = 4a \\ 2x - 3y = 7a \end{cases}$

14. $\begin{cases} 3x + y = 5a - 2b \\ x + 5a = 3y - 4b \end{cases}$

Problems

1. There are 31 pupils in an algebra class. If 3 times the number of girls exceeds twice the number of boys by 3, how many girls are there in the class? how many boys?

HINT. $g + b = 31$; $3g - 2b = 3$. Explain.

2. How much are bananas and pineapples if 8 bananas and 2 pineapples cost 60¢, while 3 bananas and 1 pineapple cost 27¢?

HINT. $8b + 2p = 60$; $3b + p = 27$. Explain.

3. Find two numbers such that their sum is 34 and twice the smaller number exceeds the larger number by 11.

HINT. $S + L = 34$; $2S - L = 11$. Explain.

4. Separate 88 into two parts such that 5 times one of the parts exceeds 4 times the other part by 8.

5. If 3 times one number increased by twice a second number is 37, but twice the first number exceeds 3 times the second number by 3, find the numbers.

6. The sum of two numbers is 4 more than twice the smaller number and $\frac{1}{4}$ of the smaller number plus $\frac{1}{5}$ of the larger number is 8. Find the numbers.

7. One half of the sum of two numbers is equal to $1\frac{1}{2}$ times their difference. Twice the larger number exceeds 3 times the smaller number by 12. What are the numbers?

8. A fruit dealer sold 3 boxes of strawberries and 2 pineapples to a customer for \$1.11, and 4 boxes of strawberries and 3 pineapples to a second customer for \$1.54. Find the price of a box of strawberries and of a pineapple.

9. A grocer paid \$55.50 for 60 doz. eggs and 70 lb. of butter. At another time he bought 75 doz. eggs and 45 lb. of butter for \$50.25. If the prices were the same each time, how much did he pay for a dozen eggs? for a pound of butter?

10. The admission tickets to an entertainment were \$.50 for adults and \$.25 for children. The receipts from the sale of 1570 tickets amounted to \$680. How many children's tickets were sold? how many tickets for adults?

11. The receipts from the sale of 500 tickets for a school entertainment were \$325. Admission tickets were \$.50 each and reserved seats were \$.25 additional. How many admission tickets were sold? how many reserved seat tickets?

12. James had 80¢ with which to buy oranges and apples. He had 4¢ too little to buy 8 oranges and 12 apples and 4¢ more than enough to buy 12 oranges and 8 apples. What was the price of the oranges? of the apples?

13. Mrs. Stevens bought 1 lb. of coffee and 5 lb. of sugar for \$.65. Later, after the coffee had increased 20% in price and the sugar had decreased $16\frac{2}{3}\%$ in price, she bought 2 lb. of coffee and 4 lb. of sugar for \$1.04. What was the first price of the coffee and of the sugar?

14. How many pounds each of 40-cent coffee and 60-cent coffee must be bought to make a mixture of 48 lb. to sell at \$.45 a pound?

15. A merchant has some tea worth \$.50 a pound and some worth \$.65 a pound. How many pounds of each must he use to make a mixture of 12 lb. worth \$.60 a pound?

16. Milk is sold at \$.16 a quart and cream at \$.72 a quart. How many quarts of each will be needed to make 18 quarts of a mixture to sell for \$6.24?

17. Mr. Drew engaged a man to work for him at \$6 a day for every day he worked, but the man was to pay \$2 a day for his board for every day he was idle. At the end of 60 days the man received \$264. How many days did the man work?

18. In sending a telegram, there is a fixed rate for the first 10 words and a fixed rate for each additional word. If a message of 31 words costs \$.98 and a message of 45 words costs \$1.40, what are the two fixed rates?

19. A man receives \$564 annually from an investment of \$10,000. If part of the investment yields $5\frac{1}{2}\%$ and the rest, 6%, how much has he invested at each rate?

20. Miss Willard invests some money at 6% and \$2000 more at 5%. If the interest on the second amount exceeds the interest on the first investment by \$55, how much has she invested at each rate?

21. A man has \$800 invested at one rate and \$1200 at a different rate. The total income is \$102 annually. If the rates of interest were interchanged, the income would be \$98. Find the rates of interest.

22. A sum at simple interest amounts to \$496 in 8 mo., and to \$510 in 15 mo. Find the sum and the rate of interest.

HINT. $P + \frac{2}{3}Pr = 496$; $P + \frac{5}{4}Pr = 510$. Eliminate the Pr term first.

23. A sum of money at simple interest amounts to \$2100 in 10 mo. and to \$2180 in 18 mo. Find the sum and the rate of interest.

24. A purse contains 18 coins — dimes and nickels. If the amount in the purse is \$1.20, how many dimes are there? how many nickels?

25. John buys 65 postage stamps for \$1. If he has 1-cent stamps and 2-cent stamps, how many has he of each?

26. If Ann gives Mary \$4, Ann will have as much money as Mary has; but if Mary gives Ann \$3, Mary will have $\frac{1}{2}$ as much as Ann then has. How much has each?

27. If Arthur gives Kenneth \$6, Kenneth will have twice as much as Arthur; but if Kenneth gives Arthur \$9, Arthur will have 3 times as much as Kenneth. How much has each?

28. The length of a rectangle exceeds its width by 3 ft. Find the dimensions if the perimeter is 26 ft.

29. The perimeter of a rectangle is 40 ft. If the length were 5 ft. less and the width 5 ft. more, the area would be 25 sq. ft. more. Find the dimensions of the rectangle.

HINT. $l + w = 20$; $(l - 5)(w + 5) = lw + 25$. Explain.

30. If a certain rectangle were 3 ft. narrower and 6 ft. longer, its area would be increased by 18 sq. ft.; but if it were 6 ft. shorter and 4 ft. wider, its area would be decreased by 36 sq. ft. What are the dimensions of the rectangle?

31. The length of a room is 8 ft. greater than its width. If each dimension is increased by 2 ft., the area will be increased by 60 sq. ft. Find the area of the floor.

32. Find two complementary angles if one of the angles is 50° larger than the other angle.

33. Find two complementary angles if 4 times one of the angles exceeds twice the other angle by 25° .

34. The difference between two supplementary angles is 40° . What is the size of each angle?

35. What is the number of degrees in each of two supplementary angles if 3 times one of the angles equals 5 times the other angle?

36. The sum of the ages of Kate and Julia is 42 yr. If 6 yr. ago Kate was $\frac{2}{3}$ as old as Julia, how old is each now?

37. Mr. Calhoun is 5 times as old as his son George. If 3 yr. ago he was 9 times as old, how old is each now?

38. Harry can row 7 mi. an hour down a river, but only 2 mi. an hour up the river. How fast can he row in still water?

HINT. Let c = the number of miles per hour of the current and H = the number of miles per hour Harry rows in still water. Then $H + c$ = the rate downstream and $H - c$ = the rate upstream. Explain.

39. A steamboat can run 9 mi. an hour upstream and 15 mi. an hour downstream. Find the rate of the boat in still water and the rate of the current.

40. A launch takes 1 hr. 36 min. to go 16 mi. down a stream, but 2 hr. 40 min. to return the same distance. What is the rate of the current? the rate of the boat in still water?

41. A picnic party went 24 mi. up a river in 4 hr. The return trip took 3 hr. What was the rate of the current and of the boat in still water?

Digit problems. The integral number 85 is an abbreviated form for $80 + 5$. Similarly, 853 is the abbreviated form of $800 + 50 + 3$. The numerals 0, 1, 2, $\dots$ 9 are called **digits**. In 853, 8 is the hundreds' digit, 5 is the tens' digit, and 3 is the units' digit.

A general number of two digits may be expressed as $10t + u$, in which expression t represents the tens' digit and u , the units' digit. If the digits are interchanged or reversed, the number is $10u + t$.

EXAMPLE. Find a number of two digits if the sum of the digits is 8 and the number formed by reversing the order is 36 less than the original number.

SOLUTION. The required equations are: $t + u = 8$ (1)

$$(10t + u) - (10u + t) = 36 \quad (2)$$

Simplify (2): $9t - 9u = 36$ (3)

Divide (3) by 9: $t - u = 4$ (4)

$$\begin{array}{r} t + u = 8 \\ t - u = 4 \\ \hline t + u = 8 \end{array} \quad (1)$$

$$\begin{array}{r} \text{Solve (1) and (4):} \\ \hline t = 6 \\ u = 2 \end{array}$$

The required number is 62.

CHECK. $6 + 2 = 8$ and $62 - 26 = 36$.

EXERCISES

Check each solution:

1. A number is expressed by two digits whose sum is 8. If the digits are interchanged, the resulting number is 18 less than the original number. Find the number.

2. A number of two digits is reversed if the sum of the digits is added to the number. The sum of the digits is 9. What is the number?

3. A number is 4 greater than 8 times the sum of its two digits. The tens' digit is 1 greater than 4 times the units' digit. Find the number.

4. A certain number is equal to 8 times the sum of its digits. The first digit exceeds the second by 5. Find the number.

5. The units' digit of a number exceeds the tens' digit by 6. If 15 is added to twice the number, the digits are interchanged. What is the original number?

6. A number is equal to 7 times the sum of its digits. If the digits are reversed, the resulting number is 36 less than the original number. What is the number?

7. A number of two digits is equal to 4 times the sum of its digits. If the digits are reversed, the new number equals 12 more than 6 times the sum of its digits. Find the number.

Number problems.

1. Two numbers are in the ratio of 3 to 5. If 3 is subtracted from the smaller number and 4 is added to the larger number, the ratio becomes 3 to 8. Find the numbers.

2. Two numbers are in the ratio of 2 to 3. If 3 is subtracted from the smaller number and 6 from the larger number, the resulting numbers are as 3 to 4. Find the original numbers.

3. A fraction in its lowest terms equals $\frac{2}{3}$. If 5 is added to the numerator, the fraction equals $\frac{3}{2}$. What is the fraction?

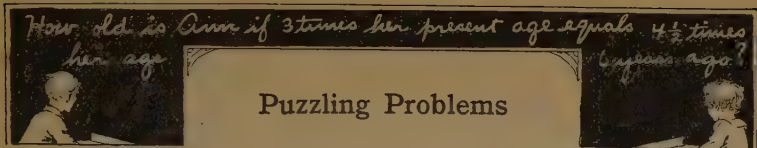
4. The numerator and the denominator of a fraction are in the ratio of 2 to 3. If 2 is added to the numerator and 6 is subtracted from the denominator, the value of the resulting fraction is $\frac{4}{3}$. Find the original fraction.

5. If 4 is added to the numerator and also to the denominator of a certain fraction, the resulting fraction equals $\frac{4}{5}$; but if 6 is subtracted from both the numerator and the denominator of the fraction, it reduces to $\frac{3}{4}$. What is the original fraction?

6. The sum of two numbers is 42. If the greater number is divided by the smaller number, the quotient is 2 and the remainder is 3. What are the numbers?

HINT. $g + s = 42$; $\frac{g - 3}{s} = 2$. Explain.

7. Divide 79 into two parts such that the quotient of the larger part divided by the other part shall be 2 with a remainder of 7.



1. In how many ways may a bill of \$23 be paid by using only \$2 and \$5 bills?

HINT. Only one equation can be written from the given conditions:

$$2d + 5n = 23$$

Solve for d :

$$d = \frac{23 - 5n}{2}$$

From the conditions given in the problem, the values of d and n must be positive and integral. By trial find the values of n that will give satisfactory values for d . Check each pair of values.

2. In how many ways may 59 be divided into two parts, one of which is divisible by 7 and the other, by 3? The quotients are to be positive integers.

3. Mrs. Martin bought coffee at \$.36 a pound and tea at \$.48 a pound, paying \$12 for the total. How many whole pounds of each did she buy?

4. Find pairs of fractions whose denominators are 16 and 24 and whose sum is $\frac{19}{24}$.

5. A miller had only 6-pound and 9-pound weights. In how many different ways could he weigh 99 lb. of grain?

6. A farmer took 100 fowl to market (turkeys, chickens, and ducks) and sold them for \$100. If he received \$3.50 for each turkey, \$1.50 for each chicken, and \$.50 for each duck, how many of each did he sell?

HINT. $t + c + d = 100$ (1)

$$3.50t + 1.50c + .50d = 100$$
 (2)

Eliminate one of the variables from (1) and (2), then proceed as in ex. 1.

7. Mr. Hunt bought 40 animals (cows, calves, and lambs) for \$480. How many of each did he buy if he paid \$40 per cow, \$16 per calf, and \$8 per lamb?

8. Mrs. Clark bought 50 yd. of dress goods for \$86. Some of it was \$1 a yard, some \$2 a yard, and the rest \$3 a yard. How many whole yards at each price did she buy?

9. A man bought 100 apples for a dollar, some at 5¢ each, some at 1¢ each, and some at $\frac{1}{2}$ ¢ each. How many of each grade did he buy?

10. A farmer took a drove of sheep and ducks to market. When asked how many of each there were, he replied that there were 100 heads and feet. How many of each were there?

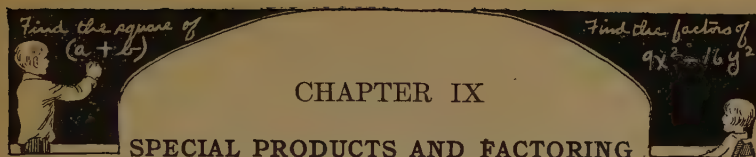
11. Mr. Hambel and Mr. Kelley were engaged to dig 100 rd. of a ditch for \$100. As the part that Mr. Kelley had to dig was harder work, he was paid \$1.40 a rod, while Mr. Hambel received \$.76 a rod. How much did each man dig?

12. A man left an estate of \$39,000 to his wife, daughter, and son on the condition that if his daughter died before coming of age, his widow was to receive $\frac{1}{4}$ of the fortune; but if the son died, the widow was to receive $\frac{3}{4}$ of the fortune. If the wife, daughter, and son all lived, how much did each receive, providing the ratio of their shares remained unchanged?

13. George agreed to carry 6 doz. eggs to market for $\frac{1}{2}$ ¢ for each egg he delivered unbroken; but he was to pay at the rate of 36¢ a dozen for each egg he broke. If he received 29¢, how many eggs did he break?

14. A boy leaving an orchard met three other boys who asked for a share of his apples. He gave $\frac{1}{2}$ to the first, who then returned 12 apples to him. He gave $\frac{1}{2}$ of the remainder to the second, who returned 7 apples. Finally he gave $\frac{1}{2}$ of what he then had to the third, who returned 4 apples. He then had 20 apples. How many apples did he have at first? How many apples did each of the other boys keep?

15. Louise and Julia went for a walk and took their lunch with them. Louise took 6 sandwiches and Julia took 4 sandwiches. Later Ruth joined them and they divided the lunch equally among the three. If Ruth paid \$.20 for her share of the lunch, how should Louise and Julia divide the \$.20?



Review of Multiplication

In Chapter IV you learned how to find the product of two algebraic expressions. In this chapter you will review the work of multiplication and will learn short methods of multiplying certain forms, as well as methods of separating an expression into its factors.

You will recall that the product of a polynomial by a monomial is equal to the sum of the products of each term of the polynomial by the monomial.

Thus, $a(b + c - d) = ab + ac - ad$.

The square of a literal term is the product of the same literal factors with their exponents multiplied by 2.

Thus, $(abc)^2 = a^2b^2c^2$ and $(ax^2y^3)^2 = a^2x^4y^6$.

The square of a monomial is the product of the square of the numerical coefficient by the square of the literal factor. The square of a monomial is always positive.

Thus, $(3a^2b^3c)^2 = 9a^4b^6c^2$; $(-5ab^2)^2 = 25a^2b^4$.

ORAL EXERCISES

Perform the indicated operations:

- | | | |
|------------------|------------------|-------------------------------|
| 1. $5(a + b)$ | 6. $(2xy)^2$ | 11. $(3x^2yz)^2$ |
| 2. $3(2a - b)$ | 7. $(3xy^2)^2$ | 12. $(15x^3y^2)^2$ |
| 3. $7(a^2 - b)$ | 8. $3(xy^2)^2$ | 13. $-3(-5a^2xy)^2$ |
| 4. $2(2a - b)$ | 9. $(-2x^3y)^2$ | 14. $(-\frac{2}{3}ax^2y^4)^2$ |
| 5. $-3a(2a - b)$ | 10. $-2(x^3y)^2$ | 15. $(2.5a^2x^2y)^2$ |

The **square root** of a number is one of the two equal factors of the number. If the square root of a number is multiplied by itself, the product is the number.

The **cube root** of a number is one of the three equal factors of the number. Similarly, the n th root of a number is one of the n equal factors of the number.

A **root** of a number is indicated by the radical sign $\sqrt{}$, with a small number, called the **index** of the root, in the opening to indicate the intended root. In the case of the square root, the index is omitted.

Thus, $\sqrt{4} = 2$, since $2 \times 2 = 4$, and $\sqrt{16 a^2 b^4} = 4 ab^2$, since $(4 ab^2)(4 ab^2) = 16 a^2 b^4$. Also, $\sqrt[3]{27} = 3$, since $3 \times 3 \times 3 = 27$.

Frequently the square root of a number or an expression cannot be exactly expressed without the aid of the radical sign. Such expressions are called **irrational expressions** or **radical expressions** and will be studied in Chapter XIII.

Thus, $\sqrt{5}$ cannot be more exactly expressed, but it has the approximate value of 2.236 . . .

In this chapter, the term *square* will refer to an expression whose square root can be exactly expressed without the aid of the radical sign.

A term that is a square is positive and the exponents of its literal factors are even numbers.

The square root of a number may be either a positive number or a negative number, since the square of a number with a negative sign is the same as the square of the same number with a positive sign. Thus, the square root of a^2 is $+a$ or $-a$.

The positive square root is called the **principal square root** and is the one required in this chapter. Accordingly, $\sqrt{a^2} = a$ and $-\sqrt{a^2} = -a$.

The process of finding the square root of a number is the reverse of squaring a number.

The square root of a term is the square root of the numerical coefficient multiplied by the literal factors with exponents half as large.

Factoring

ORAL EXERCISES

Determine which of the following terms are squares and give their principal square roots:

- | | | | |
|---------------------|---------------------------|------------------|------------------------------------|
| 1. $25 x^2$ | 4. $36 a^2 x^4$ | 7. $81 M^2 x^6$ | 10. $72 r^2 x^4 y^5$ |
| 2. $64 x^2 y^4 z^8$ | 5. $\frac{9}{16} r^2 s^8$ | 8. $121 a^4 x^2$ | 11. $196 p^4 q^{10}$ |
| 3. $16 M^2 x^8$ | 6. $225 a^2 x^9$ | 9. $169 M^4 p^6$ | 12. $\frac{222 \times 126}{169 y}$ |

Any numbers which when multiplied together give the product are called **factors** of the product.

The process of **factoring** is the reverse of multiplication, since it separates a product into parts which when multiplied together will give the product.

Thus, the factors of $3 x^2 y$ are 3, x , x , and y , also $3 x$ and xy or $3 y$ and x^2 .

If a factor cannot be separated into other factors, it is called a **prime factor**.

When the factors of an algebraic expression are required in this book, give the prime factors. It is not necessary, however, to give the prime factors of arithmetic numbers.

Since $5 \times 7 = 35$, the factors of 35 are 5 and 7. Similarly, since $a(3 a^2 + 2 a + 1) = 3 a^3 + 2 a^2 + a$, you see that a is contained in each term of $3 a^3 + 2 a^2 + a$ and the factors are a and $3 a^2 + 2 a + 1$.

Polynomials that have a *common term of the form* $ab + ac + ad$ can be factored by removing the common term.

Common monomial factor.

EXAMPLE 1. Factor $12 a + 6 b + 3 c$.

SOLUTION. The numerical factor 3 is the only factor that is common to each term of the polynomial. Remove this factor by dividing the expression by 3.

$$\frac{12 a + 6 b + 3 c}{3} = \frac{12 a}{3} + \frac{6 b}{3} + \frac{3 c}{3} = 4 a + 2 b + c.$$

There is no factor common to the terms of $4 a + 2 b + c$.

Hence $12 a + 6 b + 3 c = 3(4 a + 2 b + c)$.

EXAMPLE 2. Factor $18 a^3 b^2 + 24 a^2 b^3 + 12 ab^4$.

SOLUTION. First read the expression carefully to see if there is a common numerical factor. If there is, find the largest numerical factor contained in each of the terms. In this example, 6 is contained in 18, 24, and 12 and it is the largest number contained in each of the terms. If 6 is removed, you have left the arithmetic coefficients 3, 4, and 2. These numbers have no common factor.

Next look for all the literal factors that are common to each of the terms. It is evident that both a and b are found in each term. In $18 a^3 b^2$, a occurs three times and b twice; in $+ 24 a^2 b^3$, a occurs twice and b three times; and in $12 ab^4$, a is found once and b , four times. Hence ab^2 is the largest factor that is common to each term.

$$\text{Hence } 18 a^3 b^2 + 24 a^2 b^3 + 12 ab^4 = 6 ab^2(3 a^2 + 4 ab + 2 b^2).$$

Summary of steps.

1. Find the highest common factor of each term.
2. Divide each term by the highest common factor of the terms.
3. Write the indicated product of the highest common factor and the quotient obtained in (2) as the factors of the expression.
4. Check the work. The product of all the factors should equal the original expression.

The factors should be prime. The quotient obtained in (2) should contain no factor common to each of its terms.

EXERCISES

Factor, and check the results:

- | | |
|------------------------|-------------------------------|
| 1. $3a + 9b$ | 11. $15ac - 25bcx$ |
| 2. $4ab + 4a$ | 12. $36x^3y - 63x^2y^3$ |
| 3. $5x + ax$ | 13. $15x^3 + 10x^2 - 5x$ |
| 4. $12x^2 - 16x^3$ | 14. $16x^2 - 20xy + 12y^2$ |
| 5. $9ax^2 - 18a^2x$ | 15. $-5x^2 + 15xy - 20y^2$ |
| 6. $36x^3 - 45x$ | 16. $9a^2 - 12a + 15$ |
| 7. $\pi r^2 - 5\pi$ | 17. $-15x^2 + 10x - 5$ |
| 8. $4\pi r^2 - 2\pi r$ | 18. $3x^2 - ax + bx^3$ |
| 9. $2\pi r - 2\pi b$ | 19. $-x^3 + x^2 - x$ |
| 10. $4ax - 12bx$ | 20. $20x^3 + 25x^2y + 45xy^2$ |

21. Find the value of $15 \times 5 - 15 \times 2$.

HINT. First factor and get $15 \times 5 - 15 \times 2 = 15(5 - 2)$, or 15×3 .

Use factoring as a short cut to find the value of:

22. $20 \times 8 + 20 \times 9 - 20 \times 7$

23. $17 \times 15 - 17 \times 15 + 17 \times 10$

24. $23 \times 9 - 23 \times 3 - 23$

25. $14 \times 8 - 9 \times 14 + 14 \times 3$

26. $63 \times 8 + 4 \times 63 - 2 \times 63$

27. Factor $a(m + n) + b(m + n)$

HINT. The terms are grouped to show the common factor $m + n$. The factors are $(m + n)(a + b)$. Explain.

Factor:

28. $5(x + y) + a(x + y)$

31. $m(x - y) - n(x - y)$

29. $x(a + b) + y(a + b)$

32. $5a(x^2 - y) + (x^2 - y)$

30. $a(x + y) - b(x + y)$

33. $p(r + s) - (r + s)$

34. $ax + bx + ay + by$

SOLUTION. There is no factor common to each term of the polynomial, but x is common to ax and bx and y is common to ay and by . Hence we may write $ax + bx + ay + by = x(a + b) + y(a + b)$. This is similar to ex. 29.

Hence $ax + bx + ay + by = (a + b)(x + y)$.

35. $ax - bx + ay - by$

40. $ax + ay - bx - by$

36. $5a + ax + 5b + bx$

41. $ax - bx - ay + by$

37. $a^2 - ax + ab - bx$

42. $a^2 - ab - a + b$

38. $b - ab + 1 - a$

43. $r^3 + r^2 - 5r - 5$

39. $am - an + mx - nx$

44. $ax + 4a - 4x - 16$

45. Solve the equation $ax = ab + ac$.

SOLUTION.

$$ax = ab + ac$$

Factor the right side:

$$ax = a(b + c)$$

Divide by a :

$$x = b + c$$

CHECK. $a(b + c) = ab + ac$

Solve the following equations. Check the roots:

46. $5x = 10a + 5b$

48. $2mx + px = 4m^2 + 2mp$

47. $3x = 3p - 3q$

49. $ax = a^2 - ab + ac$

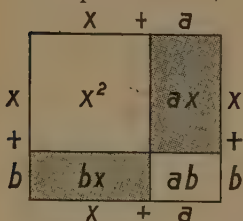
The product of two binomials of the form $(x + a)(x + b)$.

Let us review the method of finding the product of two binomials having a common term, as $(x + a)(x + b)$. (See page 62.)

$$\begin{array}{r} x + a \\ x + b \\ \hline x^2 + ax \\ + bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

Note that the product consists of three terms: the square of the x term, an x term having for a coefficient the sum of the two numbers a and b , and a term consisting of the product of the numbers a and b .

The product of $(x + a)(x + b)$ may be represented geometrically by the area of the rectangle at the left. The area of the rectangle is equal to the product of its length and width, or $(x + a)(x + b)$. It is equal also to the sum of the areas of the four smaller rectangles inclosed within the larger rectangle.



That is, $(x + a)(x + b) = x^2 + ax + bx + ab$, or $(x + a)(x + b) = x^2 + (a + b)x + ab$.

EXAMPLE 1. Find the product of $(x + 3)(x + 8)$.

SOLUTION. The first term of the result is $x \cdot x$, or x^2 .

The second term of the result is $3x + 8x = (3 + 8)x$, or $11x$.

The third term is 8×3 , or 24 .

The accuracy of the result may be checked by division or by substituting some value for x .

$$\begin{array}{r} x + 3 \\ x + 8 \\ \hline x^2 + 3x \\ + 8x + 24 \\ \hline x^2 + 11x + 24 \end{array}$$

EXAMPLE 2. Find the product of $(x + 2)(x - 5)$.

SOLUTION. $(x + 2)(x - 5) = x^2 + (2 - 5)x + 2(-5)$
 $= x^2 - 3x - 10$

The product of two binomials having a common term is the sum of the square of the common term, the product of the sum of the two unlike terms and the common term, and the product of the unlike terms.

EXERCISES

By inspection, find the indicated products. Check ex. 1 to 20:

- | | |
|----------------------------|-----------------------------|
| 1. $(x + 5)(x + 3)$ | 18. $(x + 3y)(x + 3y)$ |
| 2. $(x + 8)(x + 2)$ | 19. $(xy - 19)(xy + 3)$ |
| 3. $(x - 7)(x - 1)$ | 20. $(x^2 + 3y)(x^2 + 15y)$ |
| 4. $(a + 3)(a + 9)$ | 21. $(t - 7s)(t - 20s)$ |
| 5. $(m + 7)(m - 2)$ | 22. $(x + 8)(x + 5)$ |
| 6. $(m - 13)(m + 12)$ | 23. $(x + 8)(x - 5)$ |
| 7. $(x - 3)(x - 9)$ | 24. $(x - 8)(x + 5)$ |
| 8. $(x + 2a)(x + 3a)$ | 25. $(x - 8)(x - 5)$ |
| 9. $(x + 5b)(x - 8b)$ | 26. $(x + a)(x - b)$ |
| 10. $(a - 2b)(a + 5b)$ | 27. $(x + 2n)(x + 3n)$ |
| 11. $(n + 3)(n - 9)$ | 28. $(ab - 5c)(ab + 2c)$ |
| 12. $(xy - 5)(xy + 7)$ | 29. $(ab + 13c)(ab - 3c)$ |
| 13. $(xy - 17)(xy - 2)$ | 30. $(ab - 3x)(ab + 25x)$ |
| 14. $(x^2 - 5a)(x^2 + 3a)$ | 31. $(t + 2.4)(t - .6)$ |
| 15. $(a + 7x)(a - 9x)$ | 32. $(x - 1.2)(x + 1.1)$ |
| 16. $(x + 5a)(x + 5a)$ | 33. $(x + .5)(x - 1.8)$ |
| 17. $(x - 7a)(x - 7a)$ | 34. $(2x - .3)(2x + 1.5)$ |

By inspection, find the indicated quotients:

- | | |
|---|-------------------------------------|
| 35. $(x^2 - 5x + 6) \div (x - 2)$ | 38. $(x^2 + x - 20) \div (x + 5)$ |
| Ans. $x - 3$. Explain. | 39. $(n^2 - 3n - 40) \div (n - 8)$ |
| 36. $(x^2 - 6x + 8) \div (x - 4)$ | 40. $(n^2 - 15n + 56) \div (n - 8)$ |
| 37. $(x^2 - x - 6) \div (x - 3)$ | 41. $(n^2 + 15n + 54) \div (n + 6)$ |
| 42. $(x^2 + 2xy + y^2) \div (x + y)$ | |
| 43. $(n^2 + 15n - 54) \div (n - 3)$ | |
| 44. $(t^2 + 5st - 14s^2) \div (t - 2s)$ | |
| 45. $(t^2 - t - 2) \div (t - 2)$ | |
| 46. $(a^2 + 3ab + 2b^2) \div (a + b)$ | |
| 47. $(a^2 - 3a - 28) \div (a + 4)$ | |
| 48. $(a^2b^2 - ab - 30) \div (ab - 6)$ | |
| 49. $(x^2 - 2xy + y^2) \div (x - y)$ | |

Factors of trinomials of the form $x^2 + bx + c$. You have had considerable practice in finding products similar to $(x + 3)(x + 5) = x^2 + 8x + 15$. Now you will factor such products.

EXAMPLE 1. Find the factors of $x^2 + 8x + 15$.

SOLUTION. In factoring $x^2 + 8x + 15$, you know at once that x^2 is the product of x by x . Next try to find two factors of 15 that added together give 8. In this case, the factors of 15 whose sum is 8 are 5 and 3.

Hence the factors of $x^2 + 8x + 15$ are $(x + 5)(x + 3)$.

NOTE. The trinomial must first be arranged according to the descending or the ascending powers of some letter.

EXAMPLE 2. Factor $x^2 + 7x + 12$.

SOLUTION. First examine 12 for possible pairs of factors whose sum is 7. Consider the following pairs: 12 and 1, 6 and 2, 3 and 4. Of these three pairs of factors, the last is the only one whose sum is 7. Since 12 is positive, both factors of 12 are of the same sign, and since 7 is positive, both factors of 12 must be positive.

By trial it is shown that $x^2 + 7x + 12 = (x + 3)(x + 4)$.

EXAMPLE 3. Factor $x^2 - x - 12$.

SOLUTION. In this case 12 is negative, hence the factors of 12 are of unlike sign. Since the coefficient of x is negative, the sum of the factors of 12 is negative, hence the larger factor is negative.

By trial it is shown that $x^2 - x - 12 = (x - 4)(x + 3)$.

EXAMPLE 4. Show that the factors of $x^2 + x - 12$ are $(x + 4)(x - 3)$.

EXAMPLE 5. Show that the factors of $x^2 - 13x + 12$ are $(x - 12)(x - 1)$.

EXAMPLE 6. Show that the factors of $x^2 + 11x - 12$ are $(x + 12)(x - 1)$.

EXAMPLE 7. Show that the factors of $x^2 + 4x - 12$ are $(x + 6)(x - 2)$.

EXAMPLE 8. Show that the factors of $x^2 - 4x - 12$ are $(x - 6)(x + 2)$.

To factor $x^2 + bx + c$, find two numbers, n and m , whose product is the known term and whose algebraic sum is the coefficient of x . The factors are $(x + n)(x + m)$.

EXERCISES

Find the prime factors of:

- | | |
|----------------------|---------------------------------|
| 1. $x^2 + 5x + 6$ | 20. $x^2 - 20x + 36$ |
| 2. $x^2 + 7x + 6$ | 21. $x^2 - 27xy + 72y^2$ |
| 3. $x^2 - 5x + 6$ | 22. $x^2 + 5ax - 36a^2$ |
| 4. $x^2 - x - 6$ | 23. $x^2 + bx - 56b^2$ |
| 5. $x^2 - 5x - 6$ | 24. $x^2 - 7ax - 30a^2$ |
| 6. $x^2 + 5x - 6$ | 25. $x^2 + 14ax + 48a^2$ |
| 7. $x^2 - 6x + 8$ | 26. $a^2x^2 + 5ax - 24$ |
| 8. $x^2 + 6x + 8$ | 27. $a^2 - 2ax - 3x^2$ |
| 9. $x^2 - 2x - 8$ | 28. $a^2 + 5ab + 4b^2$ |
| 10. $x^2 + 2x - 8$ | 29. $a^2 - 7ab + 10b^2$ |
| 11. $x^2 - 9x + 8$ | 30. $a^2 + 15ax - 54a^2$ |
| 12. $x^2 + 7x - 8$ | 31. $x^2 - 3ax - 88a^2$ |
| 13. $x^2 + 12x + 27$ | 32. $a^2 + ab - 30b^2$ |
| 14. $x^2 - 9x + 18$ | 33. $a^2 + 2ab - 35b^2$ |
| 15. $x^2 + 3x - 18$ | 34. $5x^2 + 15xy + 10y^2$ |
| 16. $x^2 + 7x - 18$ | HINT. $5(x^2 + 3xy + 2y^2) = ?$ |
| 17. $x^2 + 19x + 18$ | 35. $3x^2 - 6x - 45$ |
| 18. $x^2 - 17x - 18$ | 36. $2a^2 + 16ax - 18x^2$ |
| 19. $x^2 - 13x + 36$ | 37. $4a^2 - 36a + 56$ |

$$38. -x^2 + 8x - 15$$

SOLUTION. $-x^2 + 8x - 15 = -(x^2 - 8x + 15) = -(x - 5)(x - 3)$.

The result may also be written

$$(5 - x)(x - 3) \text{ or } (x - 5)(3 - x).$$

- | | |
|-----------------------|--------------------------|
| 39. $-x^2 + 14x + 32$ | 45. $36 - 14x - 2x^2$ |
| 40. $-x^2 + 7x - 12$ | 46. $x^3 + 9x^2 - 22x$ |
| 41. $26 - 11x - x^2$ | 47. $3x^2 - 27x + 42$ |
| 42. $56 + 6x - 2x^2$ | 48. $a^4 - 2a^3 - 15a^2$ |
| 43. $77 - 4x - x^2$ | 49. $2a^2 - 28a - 30$ |
| 44. $77 + 70x - 7x^2$ | 50. $3a^2 + 12a - 63$ |

The product of two binomials of the form $(ax + b)(cx + d)$.

EXAMPLE 1. Find the product of $(3x + 4)(2x + 3)$.

SOLUTION.

$$\begin{array}{r} 3x + 4 \\ 2x + 3 \\ \hline 6x^2 + 8x \\ + 9x + 12 \\ \hline 6x^2 + 17x + 12 \end{array}$$

Notice that: (a) $6x^2$ is the product of the first two terms $(2x)(3x)$.

(b) $17x$ is the sum of the products of $2x(4)$ and $3(3x)$.

(c) 12 is the product of the last terms $(12 = 3 \times 4)$.

Since the products $8x$ and $9x$ are obtained by cross multiplying they are called **cross products**.

	$3x$	$+$	4	
$2x$	$6x^2$		$8x$	$2x$
$+$				$+$
3	$9x$		12	3
	$3x$	$+$	4	

The rectangle at the left is $(3x + 4)$ units long and $(2x + 3)$ units wide. Its area is the product of its length by its width. Examine the figure and see whether the area agrees with the

product found above, $6x^2 + 17x + 12$.

EXAMPLE 2. Find the product of $(5x - 3)(4x + 3)$.

SOLUTION.

$$\begin{array}{r} 5x - 3 \\ 4x + 3 \\ \hline 20x^2 - 12x \\ + 15x - 9 \\ \hline 20x^2 + 3x - 9 \end{array}$$

In this example, note that:

$$20x^2 = (5x)(4x)$$

$$3x = 4x(-3) + 3(5x) = -12x + 15x$$

$$-9 = 3(-3)$$

CHECK. The work may be checked by division or by numerical substitution.

If $x = 2$, then $(10 - 3)(8 + 3) = 7 \times 11 = 77$

Also, $20 \times 4 + 3 \times 2 - 9 = 80 + 6 - 9 = 77$

Summary of steps.

1. Find the product of the first terms of the binomials: $(ax)(cx) = acx^2$.
2. Find the sum of the cross products of the first terms by the second terms of the binomial: $\overbrace{(ax + b)(cx + d)} = adx + bcx = (ad + bc)x$.
3. Find the product of the second terms of the binomials: $(b)(d) = bd$.
Thus, $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$.

EXERCISES

Complete the following indicated products. Check the results:

1. $(2a + 1)(a + 3) = 2a^2 + 7a + (\quad)$
2. $(3a - 2)(a - 1) = 3a^2 - 5a + (\quad)$
3. $(5a - 1)(a + 1) = 5a^2 + 4a - (\quad)$
4. $(6a + 3)(a + 2) = 6a^2 + 15a + (\quad)$
5. $(2a - 1)(a - 5) = 2a^2 - (\quad) + 5$
6. $(3a + 2)(a + 3) = 3a^2 + (\quad) + 6$
7. $(a - 7)(2a + 3) = 2a^2 - (\quad) - 21$
8. $(5a + 3)(3a - 2) = 15a^2 - (\quad) - 6$
9. $(2x + 7y)(3x - 2y) = (\quad) + 17xy - 14y^2$
10. $(3x - y)(5x + 2y) = (\quad) + xy - 2y^2$

Find the indicated products:

- | | |
|-------------------------|----------------------------|
| 11. $(a + 3)(3a + 1)$ | 30. $(4x - 3y)(2x - 3y)$ |
| 12. $(x + 5)(3x - 1)$ | 31. $(3x - 7y)(3x + 2y)$ |
| 13. $(5x + 1)(x - 2)$ | 32. $(5ax - y)(3ax - 2y)$ |
| 14. $(3x + 1)(2x + 3)$ | 33. $(5ax + 8b)(2ax - b)$ |
| 15. $(5x - 2)(x - 3)$ | 34. $(7ax + 3b)(4ax - 2b)$ |
| 16. $(7x + 1)(2x - 3)$ | 35. $(2a + b)(3a - b)$ |
| 17. $(2a - 1)(a - 2)$ | 36. $(2a + b)(3a - 2b)$ |
| 18. $(a + 5)(3a - 11)$ | 37. $(2a + b)(3a + 5b)$ |
| 19. $(2a + 3)(a + 1)$ | 38. $(3a - 2b)(2a - b)$ |
| 20. $(a + 7)(2a + 1)$ | 39. $(5a + b)(3a + 2b)$ |
| 21. $(a - 3)(3a - 1)$ | 40. $(7a - b)(2a + 3b)$ |
| 22. $(2a - 1)(3a + 1)$ | 41. $(4a + 3b)(2a - 7b)$ |
| 23. $(2a + 1)(3a + 2)$ | 42. $(5r + 3s)(2r - 7s)$ |
| 24. $(2a - 3)(a + 4)$ | 43. $(3r - 2s)(4r + s)$ |
| 25. $(5a - 2)(2a + 3)$ | 44. $(2r + 7s)(5r - 2s)$ |
| 26. $(5x + 1)(2x + 4)$ | 45. $(.2r + 3)(.5r - 3)$ |
| 27. $(2x - 5)(5x - 2)$ | 46. $(.4a - 7)(.3a + 2)$ |
| 28. $(4x + 3)(3x - 1)$ | 47. $(.1a + 4)(.2a - 3)$ |
| 29. $(7x - y)(2x - 5y)$ | 48. $(2a + .3)(3a - .2)$ |

Factors of trinomials of the form $ax^2 + bx + c$. Many algebraic expressions are prime, just as the numbers 19, 23, 71, etc. are prime. Other expressions similar to those you have studied can be factored.

EXAMPLE 1. Factor $6x^2 + 19x + 15$.

SOLUTION. First find pairs of factors of $6x^2$ and 15 and arrange them in the form of binomials. Since the factors of $6x^2$ are $6x$ and x or $3x$ and $2x$, and the factors of 15 are 15 and 1 or 5 and 3, you may have the following possible binomials:

$$\begin{array}{r} 6x+1 \\ \times \\ x+15 \end{array} \quad \begin{array}{r} 6x+15 \\ \times \\ x+1 \end{array} \quad \begin{array}{r} 3x+15 \\ \times \\ 2x+1 \end{array}$$

$$\begin{array}{r} 3x+1 \\ \times \\ 2x+15 \end{array} \quad \begin{array}{r} 3x+3 \\ \times \\ 2x+5 \end{array} \quad \begin{array}{r} 3x+5 \\ \times \\ 2x+3 \end{array}$$

The sums of the cross products of the first terms and the last terms differ. These sums are $91x$, $21x$, $33x$, $47x$, $21x$, and $19x$. The last one is the middle term of the expression to be factored. Accordingly, you know that the factors of $6x^2 + 19x + 15$ are $3x + 5$ and $2x + 3$.

State two other possible pairs of binomial factors of $6x^2 + 19x + 15$.

At first you will find that it takes several trials before you get the correct factors; but after you have factored a number of expressions, you will find it easier. Be sure to check each set of factors by actual multiplication.

EXAMPLE 2. Factor $30x^2 + 5x - 60$.

SOLUTION. First look for a common monomial factor and, if you find one, remove it. In this expression, each term contains the factor 5. Removing the factor 5, you have $30x^2 + 5x - 60 = 5(6x^2 + x - 12)$.

The negative sign before 12 indicates that the factors of 12 are of opposite sign. Since x is positive, the larger cross product is positive. By trial you will find that $6x^2 + x - 12 = (2x + 3)(3x - 4)$.

Hence $30x^2 + 5x - 60 = 5(2x + 3)(3x - 4)$.

Note that $30x^2 + 5x - 60$ is also equal to $(10x + 15)(3x - 4)$ or $(2x + 3)(15x - 20)$; but in the first case $10x + 15$ contains the factor 5: $10x + 15 = 5(2x + 3)$. In the second case, $15x - 20 = 5(3x - 4)$.

Summary of steps.

1. Look for a monomial factor common to each of the terms. If there is one, express the product of the monomial factor and the quotient of the trinomial divided by the monomial.

2. Separate the first and last terms of the trinomial into two factors such that the algebraic sum of their cross products shall equal the middle term.

Note the sign of the last term, c . If it is $+$, each of the two factors of c has the same sign, as bx . If the sign of c is $-$, the two factors of c have opposite signs.

EXERCISES

Find the indicated factors:

1. $3x^2 + 5x + 2 = (3x + \quad)(x + \quad)$
2. $3x^2 - 5x + 2 = (3x - \quad)(x - \quad)$
3. $5x^2 - 14x - 3 = (5x + \quad)(x - \quad)$
4. $6x^2 + x - 2 = (3x + \quad)(2x - \quad)$
5. $8x^2 - 2x - 3 = (4x - \quad)(2x + \quad)$

Find the prime factors of:

- | | |
|----------------------|----------------------------|
| 6. $2x^2 - 5x + 2$ | 21. $3x^2 + ax - 10a^2$ |
| 7. $2x^2 + 5x + 2$ | 22. $6x^2 - 19ax + 10a^2$ |
| 8. $3x^2 + 7x + 2$ | 23. $6x^2 - ax - 15a^2$ |
| 9. $2x^2 + 5x + 3$ | 24. $6x^2 + ax - 5a^2$ |
| 10. $3x^2 - 2x - 5$ | 25. $8x^2 + 26xy + 15y^2$ |
| 11. $3x^2 - 10x + 3$ | 26. $15x^2 + 19xy + 6y^2$ |
| 12. $3x^2 - 8x - 3$ | 27. $10x^2 - 12y^2 + 26xy$ |
| 13. $3x^2 - x - 2$ | HINT. Rearrange the terms. |
| 14. $5x^2 + 8x + 3$ | 28. $21x^2 - 36y^2 - 51xy$ |
| 15. $5x^2 + 9x - 2$ | 29. $30x^2 - 15y^2 - 69xy$ |
| 16. $2x^2 + 5x - 3$ | 30. $42x^2 - 15y^2 + 9xy$ |
| 17. $6a^2 + 17a + 5$ | 31. $2x^2 - 21y^2 + xy$ |
| 18. $8a^2 - 10a + 3$ | 32. $10x^2 - 7y^2 - 9xy$ |
| 19. $2a^2 + a - 15$ | 33. $18x^3 + 9x^2 - 14x$ |
| 20. $3p^2 - 11p + 6$ | 34. $21a^2 + 22ab - 8b^2$ |

The square of a binomial may be found in the same way as the product of two binomials; but since the two binomials are identical, the work may be made easier by observing the following examples.

EXAMPLE 1. Find the square of $a + b$.

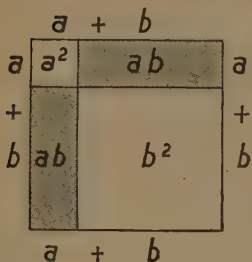
SOLUTION. This may be written $(a + b)^2$, or $(a + b)(a + b)$.

The product of the first terms is a^2

The sum of the cross products is $2ab$

The product of the last terms is b^2

Hence $(a + b)(a + b) = a^2 + 2ab + b^2$.



The square at the left is constructed on a line $(a + b)$ units long. Notice that the larger square is composed of the two smaller squares, a^2 and b^2 , and of two rectangles each of which is equal to ab square units. The complete square is equal to the square of the first term, a^2 , increased by twice the product of the first term and the second term, $2ab$, plus the square of the second term, b^2 .

EXAMPLE 2. Find the square of $2a + 3b$.

SOLUTION.

$$2a + 3b$$

$$2a + 3b$$

$$4a^2 + 6ab$$

$$6ab + 9b^2$$

$$4a^2 + 12ab + 9b^2$$

The square of $2a$ is

$$4a^2$$

Twice the product of $2a$ and $3b$ is

$$12ab$$

The square of $3b$ is

$$9b^2$$

EXAMPLE 3. Find the square of $3a - 5b$.

SOLUTION. The square of $3a$ is $9a^2$

Twice the product of $3a$ and $-5b$ is $-30ab$

The square of $-5b$ is $25b^2$

Hence $(3a - 5b)^2 = 9a^2 - 30ab + 25b^2$.

The square of the sum of two terms is equal to the square of the first term plus twice the product of the two terms plus the square of the second term.

$$(a + b)^2 = a^2 + 2ab + b^2$$

The square of the difference between two terms is equal to the square of the first term minus twice the product of the two terms plus the square of the last term.

$$(a - b)^2 = a^2 - 2ab + b^2$$

EXERCISES

Find the squares of the following binomials by inspection :

- | | | | |
|-------------|--------------|-----------------|------------------------|
| 1. $x + y$ | 7. $a - 1$ | 13. $3a + 5$ | 19. $a^2b + c^3$ |
| 2. $m + n$ | 8. $a + 2$ | 14. $7a - 2$ | 20. $7a + 3bc$ |
| 3. $p + q$ | 9. $a + 8$ | 15. $3a + 7b$ | 21. $8a - 5b^2$ |
| 4. $2x + y$ | 10. $x + 3y$ | 16. $a^2 + 1$ | 22. $a + \frac{1}{4}$ |
| 5. $x + 2y$ | 11. $a - 2y$ | 17. $a^2 + b$ | 23. $a + \frac{2}{3}$ |
| 6. $x + 3$ | 12. $m - 3n$ | 18. $x^2 - y^2$ | 24. $b - \frac{1}{2}x$ |

25. Find the square of $a + \frac{1}{2}$; of $a + 5$.

$$\begin{aligned} \text{SOLUTION. } (a + \frac{1}{2})^2 &= a^2 + a + \frac{1}{4} & (a + 5)^2 &= a^2 + 10a + 25 \\ &= a(a + 1) + \frac{1}{4} & &= a(a + 10) + 25 \end{aligned}$$

These two results are important and may be used as formulas to square numbers ending in $\frac{1}{2}$ or 5, thus :

$$(6\frac{1}{2})^2 = 6(6 + 1) + \frac{1}{4} = 6 \times 7 + \frac{1}{4} = 42\frac{1}{4}$$

$$(65)^2 = 60(60 + 10) + 25 = 60 \times 70 + 25 = 4225$$

Find by inspection the squares of the following numbers :

- | | | | |
|--------------------|---------------------|--------|---------------------|
| 26. $4\frac{1}{2}$ | 28. $12\frac{1}{2}$ | 30. 75 | 32. $13\frac{1}{2}$ |
| 27. $3\frac{1}{2}$ | 29. 55 | 31. 85 | 33. 135 |

The formula $A = p(1 + r)^t$ gives the amount of p dollars at compound interest for t years at r per cent.

34. Find $(1 + r)^t$ when $t = 2$; when $t = 3$.

35. Find $(1 + r)^2$ when $r = .04$, when $r = .05$.

36. Find A if $p = \$530$, $r = 4\frac{1}{2}\%$, and $t = 2$.

Factors of trinomials that are squares.

You have seen that: $(a + b)^2 = a^2 + 2ab + b^2$
 and $(a - b)^2 = a^2 - 2ab + b^2$

Accordingly you know that:

$$a^2 + 2ab + b^2 = (a + b)^2 \text{ and } a^2 - 2ab + b^2 = (a - b)^2$$

If a trinomial square is arranged according to the ascending or the descending powers of some common letter, the first and last terms are squares and the middle term of the trinomial is twice the product of the square roots of the first term and the last term.

ORAL EXERCISES

Complete the following trinomials so that they will be squares:

- | | |
|-------------------------|------------------------------|
| 1. $x^2 + () + y^2$ | 6. $x^2 + 6xy + ()$ |
| 2. $x^2 - () + y^2$ | 7. $9x^2 + 12xy + ()$ |
| 3. $9a^2 - () + b^2$ | 8. $16x^2 - 8x + ()$ |
| 4. $4a^2 + () + 9b^2$ | 9. $36a^4 + 84a^2 + ()$ |
| 5. $9a^2 - () + 25b^2$ | 10. $49a^2b^2 - 21abc + ()$ |

When factoring a trinomial square, find the square roots of the first and last terms and connect them with the sign of the middle term of the trinomial. There will be two binomial factors that are identical.

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$

In factoring trinomials, inspect the expression carefully to see if it is a square.

ORAL EXERCISES

Determine which of the following trinomials are squares and factor as many as possible:

- | | |
|-------------------------|-----------------------------|
| 1. $a^2 + 4ab + 4b^2$ | 8. $a^2 + 17ab + 16b^2$ |
| 2. $a^2 + 5ab + 4b^2$ | 9. $a^2 - 15ab - 16b^2$ |
| 3. $a^2 - 3ab - 4b^2$ | 10. $9a^2 + 12ab + 4b^2$ |
| 4. $a^2 + 3ab - 4b^2$ | 11. $9a^2 - 12ab + 4b^2$ |
| 5. $a^2 + 8ab + 16b^2$ | 12. $9a^2 - 15ab + 4b^2$ |
| 6. $a^2 + 10ab + 16b^2$ | 13. $9a^2 + 37ab + 4b^2$ |
| 7. $a^2 - 10ab + 16b^2$ | 14. $4a^2 - 20ab^2 + 25b^4$ |

EXAMPLE. Find the factors of $36x^2 + 84xy + 49y^2$.

SOLUTION. $36x^2$ and $49y^2$ are squares and $84xy$ is twice the product of $(\sqrt{36x^2})(\sqrt{49y^2})$, since $84xy = 2(6x)(7y)$.

Hence $36x^2 + 84xy + 49y^2 = (6x + 7y)^2$, or $(6x + 7y)(6x + 7y)$.

EXERCISES

Find the factors of the following trinomials. Several of them are not squares:

- | | |
|---|--------------------------------|
| 1. $4x^2 + 4xy + y^2$ | 16. $25x^4 - 70x^2y^3 + 49y^6$ |
| 2. $x^2 + 4xy + 4y^2$ | 17. $4a^2 + 16ab + 16b^2$ |
| 3. $9x^2 + 6xy + y^2$ | 18. $81p^2 + 108pq + 36q^2$ |
| 4. $9x^4 + 12x^2y + 4y^2$ | 19. $16m^2 - 96mp + 144p^2$ |
| 5. $16x^2 - 8x + 1$ | 20. $75x^2 - 60xy + 12y^2$ |
| 6. $25x^2 + 10x + 1$ | 21. $45a^2 - 120ab + 80b^2$ |
| 7. $25x^2 + 40x + 16$ | 22. $36a^2 - 100a + 25$ |
| 8. $25x^2 - 30x + 9$ | 23. $36a^2 + 109ab + 25b^2$ |
| 9. $49x^2y^2 - 14xy + 1$ | 24. $36a^2 + 60ab + 25b^2$ |
| 10. $81x^4 - 126x^2y + 49y^2$ | 25. $36a^2 + 61ab + 25b^2$ |
| 11. $16a^2 + 72ab^2 + 81b^4$ | 26. $64x^2 - 48xy + 9y^2$ |
| 12. $4x^2 + 2x + \frac{1}{4}$ | 27. $64x^2 - 52xy + 9y^2$ |
| 13. $9a^2 - 2ab + \frac{1}{9}b^2$ | 28. $64x^2 + 50xy + 9y^2$ |
| 14. $4m^2 - \frac{4}{3}m + \frac{1}{9}$ | 29. $64a^2 - 160ab + 100b^2$ |
| 15. $9x^2 - 3x + .25$ | 30. $144x^6 - 216x^3y + 81y^2$ |

Solve the following equations:

31. $(a - 2)x = a^2 - 4a + 4$

Ans. $x = a - 2$. Explain.

32. $5x + ax = 25 + 10a + a^2$

33. $3x - 9 = 6p + p^2 - px$

34. $ax - c^2 = a^2 + cx - 2ac$

35. $ax + a = a^2 + \frac{1}{2}x + \frac{1}{4}$

36. $4ab - bx = 4a^2 - 2ax + b^2$

37. $b^2 + 3x = b + bx + 6$

38. $ax - a = a^2 - 3x - 6$

The product of the sum of two numbers by their difference.

EXAMPLE. Find the product of $(a + b)(a - b)$.

SOLUTION.

$a + b$	The product of the first terms is	a^2
$a - b$	The sum of the cross products is	0
$a^2 + ab$	The product of the last terms is	$-b^2$
$-ab - b^2$	Hence $(a + b)(a - b) = a^2 - b^2$.	
$a^2 \quad - b^2$		

The product of the sum of two numbers by their difference is equal to the square of the first number minus the square of the second number.

$$(a + b)(a - b) = a^2 - b^2$$

EXERCISES

Find the indicated products:

- | | | |
|------------------------------|--------------------------------|--------------------|
| 1. $(a + 5)(a - 5)$ | 17. $(a^2 + b^2)(a^2 - b^2)$ | |
| 2. $(x + 3)(x - 3)$ | 18. $(x^3 + y^3)(x^3 - y^3)$ | |
| 3. $(a + x)(a - x)$ | 19. $(2.5x - 3)(2.5x + 3)$ | |
| 4. $(a + 9)(a - 9)$ | 20. $(3.5m + n)(3.5m - n)$ | |
| 5. $(a + 6)(a - 6)$ | 21. $(4.5x - 2y)(4.5x + 2y)$ | |
| 6. $(2a + b)(2a - b)$ | 22. $(7rs + 8p)(7rs - 8p)$ | |
| 7. $(r + 3s)(r - 3s)$ | 23. $(.5x - 3y)(.5x + 3y)$ | |
| 8. $(2a + 5b)(2a - 5b)$ | 24. $(55x + 3y^2)(55x - 3y^2)$ | |
| 9. $(a^2 + b)(a^2 - b)$ | 25. $5(7 - x)(x + 7)$ | |
| 10. $(7ab - c)(7ab + c)$ | 26. $\pi(2R + r)(2R - r)$ | |
| 11. $(2x^2y + 1)(2x^2y - 1)$ | 27. $2\pi(R - 2r)(R + 2r)$ | |
| 12. $(10a - b)(10a + b)$ | 28. $7(2x - 3y)(2x + 3y)$ | |
| 13. $(3xy - 2z)(3xy + 2z)$ | 29. $7(2x - 3y)(2x - 3y)$ | |
| 14. $(8m + 3p)(8m - 3p)$ | 30. $7(2x + 3y)(2x + 3y)$ | |
| 15. $(15abc - 5)(15abc + 5)$ | 31. $(20 + 2)(20 - 2)$ | |
| 16. $(5a + 2x)(5a - 2x)$ | 32. $(30 - 3)(30 + 3)$ | |
| 33. 19×21 | 36. 22×38 | 39. 85×75 |
| 34. 91×89 | 37. 73×67 | 40. 25×35 |
| 35. 22×18 | 38. 35×45 | 41. 46×54 |

Factors of binomials that are the difference of two squares.

Since $(a + b)(a - b) = a^2 - b^2$, you know that:

$$a^2 - b^2 = (a + b)(a - b)$$

The factors of a binomial that is the difference between two squares are the sum of the square roots of the terms and the difference of the square roots of the terms.

$$a^2 - b^2 = (a + b)(a - b)$$

EXERCISES

Factor the following binomials. Do ex. 1 to 15 orally:

- | | | |
|----------------|--------------------|-------------------------|
| 1. $a^2 - 1$ | 6. $9p^2 - 16q^2$ | 11. $48a^4 - 75$ |
| 2. $a^2 - 4$ | 7. $9p^2 - 144q^2$ | 12. $100a^6 - 324x^2$ |
| 3. $a^2 - 9$ | 8. $16m^2 - 25p^2$ | 13. $108a^3 - 147ab^2$ |
| 4. $a^2 - 16$ | 9. $16m^2 - 64n^2$ | 14. $169x^2 - 225y^2$ |
| 5. $x^2 - y^2$ | 10. $25x^2 - 9y^4$ | 15. $1.69x^2 - 2.25y^2$ |

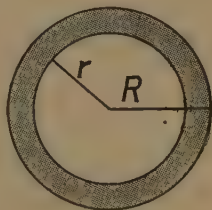
Find the value of:

- | | | |
|-------------------|---|-------------------|
| 16. $11^2 - 9^2$ | SOLUTION. $(11 + 9)(11 - 9) = 20 \times 2 = 40$ | |
| 17. $18^2 - 8^2$ | 19. $25^2 - 15^2$ | 21. $35^2 - 15^2$ |
| 18. $15^2 - 12^2$ | 20. $28^2 - 8^2$ | 22. $75^2 - 25^2$ |

Factor:

- | | |
|-----------------|--|
| 23. $a^4 - b^4$ | SOLUTION. $(a^2 + b^2)(a^2 - b^2) = (a^2 + b^2)(a + b)(a - b)$ |
| 24. $16x^4 - 1$ | 26. $16x^4 - 81y^4$ |
| 25. $81a^4 - 1$ | 27. $625 - a^4$ |
| | 28. $a^8 - b^8$ |
| | 29. $256x^8 - 1$ |

30. The area of the flat ring at the right is given by the formula $\pi R^2 - \pi r^2$. Explain how this formula is obtained. Show how the formula may be put into a more convenient form for work by factoring.



31. Find the area of the ring if $\pi = 3\frac{1}{7}$, $R = 10$, and $r = 8$.

The product of two trinomials of the form $(a + b) + c$ and $(a + b) - c$.

EXAMPLE. Find the product of $[(x + y) - z][(x + y) + z]$.

SOLUTION. This comes under the type form $(a + b)(a - b)$. The result is $(x + y)^2 - z^2$, or $x^2 + 2xy + y^2 - z^2$.

EXERCISES

Find the products:

1. $[(a + b) + c][(a + b) - c]$
2. $[(2a - b) + c][(2a - b) - c]$
3. $[(m + n) - p][(m + n) + p]$
4. $(x - y + z)(x - y - z)$
5. $[x - (y + z)][x + (y + z)]$
6. $[3a - (b + 2c)][3a + (b + 2c)]$
7. $(2a - b + c)(2a + b - c)$
8. $(x - y - 2z)(x + y + 2z)$
9. $(2x + y + z)(2x + y - z)$
10. $(2x - y - z)(2x + y - z)$
11. $(m + 3n - 2)(m + 3n + 2)$
12. $(m - 2x + 7y)(m + 2x - 7y)$
13. $(p - 7q + t)(p - 7q - t)$
14. $(3r - 2s + 1)(3r + 2s - 1)$

Factors of expressions of the form $(a + b)^2 - c^2$.

EXAMPLE 1. Find the factors of $(x + y)^2 - z^2$.

SOLUTION. This is of the form $a^2 - b^2$, in which $(x + y)$ is a and z is b .

$$a + b \quad a - b$$

The factors are $[(x + y) + z][(x + y) - z]$, or $(x + y + z)(x + y - z)$.

EXAMPLE 2. Find the factors of $x^2 - (y + z)^2$.

$$a + b \quad a - b$$

SOLUTION. The factors are $[x + (y + z)][x - (y + z)]$, or $(x + y + z)(x - y - z)$.

Keep the parenthesis around the second terms in the first step to avoid making mistakes in signs. When you remove the parenthesis, be careful to change the signs if the parenthesis is preceded by a minus sign.

EXERCISES

Find the factors of the following expressions :

1. $(x + y)^2 - 4$
2. $(x - y)^2 - 9$
3. $(2x + y)^2 - 25$
4. $4(x + y)^2 - 121z^2$
5. $9(3x - y)^2 - 16z^2$
6. $16 - (2x + y)^2$
7. $9x^2 - 4(3x - y)^2$
8. $25 - (x - 2y)^2$
9. $25a^2 - 64(x - 2y)^2$
10. $9a^2 - (2b + 3c)^2$
11. $(a + b)^2 - (c + d)^2$
12. $(m + 3)^2 - (p - 2)^2$
13. $(2a - c)^2 - (b - d)^2$
14. $9(a + 5c)^2 - (2x + y)^2$
15. $16(a - 3b)^2 - 9(3x - a)^2$
16. $(m + p)^4 - 25(5 + 2s)^2$

$$17. 9x^2 - 6xy + y^2 - 25z^2$$

$$\begin{aligned} \text{HINT. } 9x^2 - 6xy + y^2 - 25z^2 &= (9x^2 - 6xy + y^2) - 25z^2 \\ &= (3x - y)^2 - (5z)^2 \end{aligned}$$

Proceed as in example 1, page 174.

18. $a^2 - 2ab + b^2 - c^2$
19. $a^2 + 14a + 49 - 9b^2$
20. $x^2 + 4xy + 4y^2 - 1$
21. $9x^2 - 6xy + y^2 - 25$
22. $a^2 + 10ab + 25b^2 - 49c^2$
23. $a^2 - 2ab + b^2 - 16c^2$

$$24. 25a^2 + 10ab + b^2 - 81p^2$$

$$25. 16p^2 - 40pq + 25q^2 - 36m^2$$

$$26. 25x^2 - 49y^2 + 14yz - z^2$$

$$\begin{aligned} \text{HINT. } 25x^2 - 49y^2 + 14yz - z^2 &= 25x^2 - (49y^2 - 14yz + z^2) \\ &= (5x)^2 - (7y - z)^2. \end{aligned}$$

Proceed as in example 2, page 174.

27. $25 - 4x^2 - 4xy - y^2$
28. $9x^2 - 4y^2 + 12yz - 9z^2$
29. $16x^2 - 49a^2 - 14ab - b^2$
30. $a^2 - b^2 - 25 + 10b$
31. $b^2 - x^2 - y^2 + 2xy$
32. $9x^2 - a^2 - b^2 - 2ab$
33. $bc^2 - 9a^2b - b^3 - 6ab^2$
34. $a^2 - 2ax + x^2 - 16b^2$
35. $25c^2 - 9x^2 - y^2 + 6xy$
36. $xy^2 - 4x^3 - 12x^2z - 9xz^2$

$$37. 3a^3 + 3ab^2 - 3ac^2 - 6a^2b.$$

$$38. 100c^2 - 36b^2 + 24ab - 4a^2$$

$$39. (3x + 2y)^2 - (x + 2)^2$$

$$40. (2a + b)^2 - (2c - d)^2$$

$$41. x^2 + y^2 - m^2 - 25 - 2xy + 10m$$

Summary of Factoring

No general rule can be given that will enable you to know how to factor all expressions, but the following suggestions will be helpful.

Read the expression carefully to determine its type form. Always make sure that you have found the prime factors.

Look first for a monomial factor that is common to each term of the expression. If there is such a factor, remove it.

Summary of type forms.

1. *Monomial factors common to each of the terms.*

$$ab + ac + ad$$

2. *Trinomials of the general type.*

$$ax^2 + bx + c$$

Two special cases may be grouped with this type:

- (a) Trinomials having the coefficient of x^2 unity.

$$x^2 + bx + c$$

- (b) Trinomials that are squares.

$$a^2 \pm 2ab + b^2$$

3. *Binomials that are the difference of two squares.*

$$a^2 - b^2$$

- (a) Polynomials that may be grouped to show the difference between two squares.

$$a^2 + 2ab + b^2 - c^2$$

REVIEW EXERCISES

Find the prime factors of:

1. $a^2 - a - 30$

9. $10x^2 + 13x - 3$

2. $1 - 16x^4$

10. $4a^2 - a - 14$

3. $a^5 - a$

11. $m^2 - 11m + 18$

4. $x^2 + x + \frac{1}{4}$

12. $a^4 - a^2$

5. $18 - 50b^4$

13. $6x^2 - x - 2$

6. $6x^2 - 20x + 6$

14. $97^2 - 9$

7. $x^4 - 8x^2 - 9$

15. $x^{2n} + 2x^ny^n + y^{2n}$

8. $16 - x^4$

16. $162 - 2p^2$

17. $2a^2b - 6ab - 20b$
18. $x^2y^2 - 11xyz - 42z^2$
19. $12 - 10a - 2a^2$
20. $6x^2 - 9x - 6$
21. $6a^2 - 25a - 91$
22. $4a^3b - 4ab^2$
23. $16a - 25ab^2$
24. $x^2 + 2ax$
25. $18a^2 + 3a - 36$
26. $m^2 + 7 - 8m$
27. $8 - 18x^2$
28. $6a^2 + 7a + 2$
29. $5a^2 - 5a - 60$
30. $2a^2 - ab - b^2$
31. $x^2y^2 - xy^2 - 6y^2$
32. $2a^2 + 5ab - 3b^2$
33. $16a^2 + 25b^2 + 40ab$
34. $3a^2 - a - 2$
35. $2a^2b - 6ab + 18b$
36. $4a^2 - 12ab + 9b^2$
37. $a^2b^2 - 12abx + 35x^2$
38. $49a^2 - 42ab + 9b^2$
39. $19 \times 3 - 8 \times 19 + 13 \times 19$
40. $2a^6 - 10a^4 - 28a^2$
41. $103^2 - 9$
42. $m^2 - 6mn - 16x^2y^2 + 9n^2$
43. $(2a - b)x - y(2a - b)$
44. $2x(a - 2b) - 3y(a - 2b)$
45. $c(2 - x) + (2 - x)ab$
46. $a^2 - 49b^2 - 10a + 25$
47. $(a - 2b)^2 - (c - 3d)^2$
48. $(p - q)^2 - 7(p - q) + 12$
49. $(a - x)^2 - 3(a - x) + 2$
50. $12ab^3 - 12ab$
51. $a^2 + b^2 + 2ab - c^2$
52. $9a^2 - 6a - 49b^2 + 1$
53. $5x^{2n} - 5x^n + 60$
54. $3a^{2n} + 13a^n + 4$
55. $8x^{2n} - 21x^ny^n - 9y^{2n}$
56. $77 - 30x^{2n} - 37x^n$
57. $16x^{4a} - 12x^{2a} - 70$
58. $a^{2n} - 25a^nb^x + 136b^{2x}$
59. $(a - b)^2 - 6(a - b) + 9$
60. $3x^8 + 6x^4 - 9$
61. $a^2 + .2a - .15$
62. $x^2 - 2.9x - .3$
63. $2x^2 + .3x - .02$
64. $x^4 - 13x^2 + 36$
65. $x^3 - 2x^2 - 15x$
66. $x^4 - 10x^2 + 9$

Solve the following equations:

67. $px - 4qx = 5p^2 - 23pq + 12q^2$
68. $3ax + 5bx = 21a^2 - 10b^2 + 29ab$
69. $2ax + 5 = 12a^4 + 17a^2 + x$
70. $mx + 13mn = 8m^2 + 5n^2 + nx$
71. $3mx + 18mn = 9m^2 + 4nx + 8n^2$
72. $5ax + 49b^2 = 25a^2 - 7bx$
73. $3.5ax - 4bx = 12.25a^2 - 28ab + 16b^2$
74. $5px - 3qx = 5p^2 + 17pq - 12q^2$

How old is Ann if 3 times her present age equals $4\frac{1}{2}$ times her age 4 years ago?



Puzzling Problems



1. If $32 \text{ ounces} = 2 \text{ pounds}$ (1)

and $8 \text{ ounces} = \frac{1}{2} \text{ pound}$ (2)

Then, $(1) \times (2)$, $256 \text{ ounces} = 1 \text{ pound}$

since if equals are multiplied by equals the results are equal.
Can you find the error in this reasoning?

2. An egg man sold half of his eggs and half an egg to Mrs. Smith, half of the remaining eggs and half an egg to Mrs. Bell, and half of the eggs then left and half an egg to Mrs. Brown. Mrs. Potter took all the rest of the eggs and had 36 eggs. How many eggs were there at first?

3. Let $n = 1$

Then $n^2 = n$ Why?

and $n^2 - 1 = n - 1$ Why?

$n + 1 = 1$ Why?

But $n = 1$

Then, by substitution, $2 = 1$ Explain the error.

4. Explain this fallacy:

$$8 - 8 = 2 - 2, \text{ as each} = 0$$

and $2 - 2 = 2 - 2$

Then $4 = 1$

For $8 - 8 = 4(2 - 2)$, which equals $2 - 2$

Divide by $2 - 2$: $4 = 1$

Or, all numbers are equal to each other.

5. Explain this fallacy:

Let $a = b$

Then $ab = b^2$ and $a^2 = b^2$

$$ab - b^2 = a^2 - b^2$$

$$b(a - b) = (a + b)(a - b)$$

$$b = a + b = b + b$$

Or $b = 2b$

Hence $1 = 2$



CHAPTER X

FRACTIONS



You have worked a number of exercises that contained fractions with numerical denominators, but you will need to know how to handle fractions that have algebraic expressions in the denominator as well as in the numerator.

A *fraction* is an indicated division. (See page 6.) The dividend is the numerator of the fraction and the divisor is the denominator.

Thus, $\frac{a}{b}$ is a fraction. a and b represent any algebraic expressions.

The rules that apply to fractions in arithmetic apply also to fractions in algebra.

Reducing Fractions to Lowest Terms

Fractions are in their **lowest terms** when their numerators and denominators are prime to each other, that is, when they contain no common factors except 1.

Thus, $\frac{12}{18}$ is reduced to $\frac{2}{3}$ by dividing the numerator and the denominator of the fraction by the factor 4 that is common to both terms.

EXAMPLE. Reduce $\frac{4a^2b}{6ab^2}$ to lowest terms.

SOLUTION. Factor both terms of the fraction. Then divide both terms of the fraction by all factors that are common to both.

$$\frac{4a^2b}{6ab^2} = \frac{2 \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{a}} \overset{1}{\cancel{b}}}{3 \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{2}} \times \overset{1}{\cancel{a}} \overset{1}{\cancel{b}}} = \frac{2a}{3b}$$

Note that $\frac{\overset{1}{\cancel{a}}}{\underset{1}{\cancel{a}}} = \frac{1}{1} = 1$. If a factor in the denominator is

divided into a like factor of the numerator, the quotient is 1. If the numerators and denominators cancel, the result is 1.

The reduction of fractions to lowest terms depends on the principle that:

Both terms of a fraction may be divided by the same factors without changing the value of the fraction.

EXERCISES (1 to 12 Oral)

Reduce to lowest terms:

- | | | |
|----------------------|----------------------------------|---------------------------------------|
| 1. $\frac{12}{15}$ | 11. $\frac{8ab}{12a}$ | 21. $\frac{35a^3b^2x}{-42ab^2x^2}$ |
| 2. $\frac{18}{27}$ | 12. $\frac{9pq}{12p}$ | 22. $\frac{48xy^2z^3}{64x^3y^2z}$ |
| 3. $\frac{12}{32}$ | 13. $\frac{6a^2b}{15ab^2}$ | 23. $-\frac{32pq}{36p^2}$ |
| 4. $\frac{14}{21}$ | 14. $\frac{8ab^3}{18a^2b}$ | 24. $\frac{21axy}{18ay}$ |
| 5. $-\frac{42}{56}$ | 15. $\frac{15xy^2}{18x^2y^2}$ | 25. $\frac{15x^2y}{15xy^2}$ |
| 6. $-\frac{18}{24}$ | 16. $\frac{14abc}{35a^2c}$ | 26. $\frac{9a^2b}{18a^2b}$ |
| 7. $-\frac{9}{15}$ | 17. $-\frac{21x^3y^2}{42x^2y^3}$ | 27. $\frac{25x^3y^2}{52xy^3}$ |
| 8. $-\frac{16}{24}$ | 18. $-\frac{16a^2bc}{24ab^2c}$ | 28. $\frac{18p^2q^2}{30pqz}$ |
| 9. $-\frac{72}{45}$ | 19. $-\frac{32a^3b}{24ab^3}$ | 29. $\frac{a^nb^2}{a^3b^5}$ |
| 10. $-\frac{27}{18}$ | 20. $\frac{48x^3y^4}{40y^2z^3}$ | 30. $\frac{12a^5nx^{3n}}{8a^nx^{4n}}$ |

EXAMPLE 1. Reduce $\frac{x^2 - y^2}{x^2 + 2xy + y^2}$ to lowest terms.

SOLUTION. Factor both terms of the fraction. Then divide both terms of the fraction by the factors that are common to each term.

$$\frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{\overset{1}{(x+y)}(x-y)}{\underset{1}{(x+y)}(x+y)} = \frac{x-y}{x+y}$$

CHECK. Let $x = 2$ and $y = 3$

Then
$$\frac{x^2 - y^2}{x^2 + 2xy + y^2} = \frac{4 - 9}{4 + 12 + 9} = \frac{-5}{25} = -\frac{1}{5}$$

Also
$$\frac{x - y}{x + y} = \frac{2 - 3}{2 + 3} = -\frac{1}{5}$$

EXAMPLE 2. Reduce $\frac{3a^2 - 3ab - 18b^2}{12a^2 - 18ab - 54b^2}$ to lowest terms.

SOLUTION. Factor both terms of the fraction and divide by the common factors.

$$\frac{3a^2 - 3ab - 18b^2}{12a^2 - 18ab - 54b^2} = \frac{\overset{1}{3}(a + 2b)(\overset{1}{a} - 3b)}{\underset{2}{6}(2a + 3b)(\underset{1}{a} - 3b)} = \frac{a + 2b}{2(2a + 3b)}$$

Check by substitution.

Finding the highest common factor.

In arithmetic, as you will remember, the product of all the prime factors common to two or more numbers is called their *greatest common divisor* (G. C. D.).

Thus, since $27 = 3 \times 3 \times 3$ and $36 = 3 \times 3 \times 4$, the G. C. D. of 27 and 36 is 3×3 , or 9.

In algebra the product of all the prime factors that are common to two or more numbers is called their **highest common factor** (H. C. F.).

Thus, $3(a - 3b)$ is the H. C. F. of $3a^2 - 3ab - 18b^2$ and $12a^2 - 18ab - 54b^2$. Fractions can be reduced to their lowest terms by dividing the numerator and the denominator by their H. C. F.

NOTE. The fraction $\frac{2a - b}{3a + b}$ cannot be reduced to lower terms, as there is no factor common to its numerator and denominator. You can see that it would be wrong to cancel numbers that look alike.

Thus,
$$\frac{\overset{1}{2}\cancel{a} - \overset{1}{b}}{\overset{1}{3}\cancel{a} + \overset{1}{b}}$$
 does not equal $\frac{2 - 1}{3 + 1}$, or $\frac{1}{4}$

If you substitute numbers, as 10 for a and 5 for b ,

$$\frac{2a - b}{3a + b} = \frac{2 \times 10 - 5}{3 \times 10 + 5} = \frac{15}{35} = \frac{3}{7} \text{ and } \frac{3}{7} \text{ does not equal } \frac{1}{4}.$$

EXERCISES

Reduce to lowest terms. Check ex. 1 to 9:

$$1. \frac{ab}{ab + ac}$$

$$2. \frac{ab}{ab + bc}$$

$$3. \frac{bc}{b^2 + b}$$

$$4. \frac{3a}{3a^2 + 3ab}$$

$$5. \frac{abc}{a^2 + ab}$$

$$6. \frac{3ab + 6b^2}{a^2 - 4b^2}$$

$$7. \frac{x^2 + xy}{x^2 - xy}$$

$$8. \frac{a^2 + ab}{ab + b^2}$$

$$9. \frac{x - y}{x^2 - y^2}$$

$$10. \frac{a^2 + ab}{a^2 - b^2}$$

$$11. \frac{a^2 - b^2}{a^2 - 3ab + 2b^2}$$

$$12. \frac{x^2 + xy}{x^3 + 2x^2y + xy^2}$$

$$13. \frac{x^2 - 9y^2}{x^2 - 6xy + 9y^2}$$

$$14. \frac{a^2 + 2ab + b^2}{a^2 + 3ab + 2b^2}$$

$$15. \frac{x^2 + 7xy + 12y^2}{x^2 - 16y^2}$$

$$16. \frac{4x^2 + 4xy + y^2}{2x^2 + 5xy + 2y^2}$$

$$17. \frac{x - y}{y - x}$$

$$\text{HINT. } \frac{x - y}{y - x} = \frac{x - y}{-(x - y)}$$

$$18. \frac{2x - y}{y^2 - 4x^2}$$

$$19. \frac{2a - 3b}{9b^2 - 4a^2}$$

$$20. \frac{6a^2 + 7ab - 3b}{3a^2 - 7ab + 2b^2}$$

$$21. \frac{3x^2 + 9xy}{x^2 + xy - 6y^2}$$

$$22. \frac{3a^2 - 11a + 10}{6 + 9a - 6a^2}$$

$$23. \frac{2ax - ay}{16x^2 - 4y^2}$$

$$24. \frac{36a^2 - 9}{12a^2 + 12a - 9}$$

$$25. \frac{4\pi R^2 - 9\pi}{2\pi R^2 + 3\pi R}$$

$$26. \frac{x^3 - 6x^2 + 8x}{6x^4 - 24x^2}$$

$$27. \frac{2x^2 - 5x + 2}{3x^2 - 12x + 12}$$

$$28. \frac{a^4 - 13a^2 + 36}{a^2 - a - 6}$$

$$29. \frac{a^4 - 8a^2 + 16}{a^2 + 4a + 4}$$

$$30. \frac{a^2 + 2ab + b^2 - c^2}{a^2 - b^2 + 2bc - c^2}$$

$$31. \frac{x^2 - y^2 - 2yz - z^2}{x^2 - xy - xz}$$

Multiplying Fractions

Fractions are multiplied in algebra in the same way as in arithmetic.

$$\begin{aligned}\text{Thus,} \quad \frac{2}{3} \times \frac{4}{5} &= \frac{2 \times 4}{3 \times 5} = \frac{8}{15} \\ \frac{a}{b} \times \frac{c}{d} &= \frac{a \times c}{b \times d} = \frac{ac}{bd}\end{aligned}$$

EXAMPLE 1. Multiply $\frac{3a}{14b}$ by $\frac{35b^2}{9a^3}$.

$$\text{SOLUTION.} \quad \frac{3a}{14b} \times \frac{35b^2}{9a^3} = \frac{\overset{1}{\cancel{3}a}}{\underset{2}{14}b} \times \frac{\overset{5b}{\cancel{35}b^2}}{\underset{3a^2}{9a^3}} = \frac{5b}{6a^2}$$

CHECK. Let $a = 2$, $b = 3$

$$\text{Then} \quad \frac{3a}{14b} \times \frac{35b^2}{9a^3} = \frac{3(2)}{14(3)} \times \frac{35(3)^2}{9(2)^3} = \frac{6}{42} \times \frac{315}{72} = \frac{5}{8}$$

$$\text{Also} \quad \frac{5b}{6a^2} = \frac{5(3)}{6(2)^2} = \frac{15}{24} = \frac{5}{8}$$

The factor $3a$ divides into the first numerator with a quotient of 1, and into the second denominator with a quotient of $3a^2$. The factor $7b$ divides into the first denominator with a quotient of 2, and into the second numerator with a quotient of $5b$. The resulting fraction is found by taking the product of the remaining factors of the numerators for the numerator, and the product of the remaining factors of the denominators for the denominator.

EXAMPLE 2. Multiply $\frac{a^2 - b^2}{a^2 - 3ab + 2b^2}$ by $\frac{a^2 - 4ab + 4b^2}{a^2 + ab}$.

SOLUTION

$$\begin{aligned}\frac{a^2 - b^2}{a^2 - 3ab + 2b^2} \times \frac{a^2 - 4ab + 4b^2}{a^2 + ab} &= \frac{\overset{1}{\cancel{(a+b)}}(\overset{1}{\cancel{(a-b)}})}{\underset{1}{\cancel{(a-2b)}}(\underset{1}{\cancel{(a-b)}})} \times \frac{\overset{1}{\cancel{(a-2b)}}(\overset{1}{\cancel{(a-2b)}})}{\underset{1}{a}(\underset{1}{\cancel{(a+b)}})} \\ &= \frac{a - 2b}{a}\end{aligned}$$

CHECK. Let $a = 2$, $b = 3$

$$\begin{aligned}\text{Then} \quad \frac{a^2 - b^2}{a^2 - 3ab + 2b^2} \times \frac{a^2 - 4ab + 4b^2}{a^2 + ab} &= \frac{4 - 9}{4 - 18 + 18} \times \frac{4 - 24 + 36}{4 + 6} \\ &= \frac{-5}{4} \times \frac{16}{10} = -2\end{aligned}$$

$$\text{Also} \quad \frac{a - 2b}{a} = \frac{2 - 6}{2} = -2$$

EXAMPLE 3. Multiply $a^2 - 16 b^2$ by $\frac{a^2 + 3 ab - 4 b^2}{a^2 - 5 ab + 4 b^2}$.

SOLUTION. Write the integral expression, $a^2 - 16 b^2$, as a fraction with denominator 1.

$$\begin{aligned} \text{Then } \frac{a^2 - 16 b^2}{1} \times \frac{a^2 + 3 ab - 4 b^2}{a^2 - 5 ab + 4 b^2} \\ = \frac{\overset{1}{(a-4b)}(a+4b)}{1} \times \frac{(a+4b)\overset{1}{(a-b)}}{\underset{1}{(a-4b)}(\underset{1}{a-b})} = (a+4b)^2 \end{aligned}$$

CHECK. Let $a = 2$, $b = 3$

$$\begin{aligned} \text{Then } \frac{a^2 - 16 b^2}{1} \times \frac{a^2 + 3 ab - 4 b^2}{a^2 - 5 ab + 4 b^2} &= \frac{4 - 144}{1} \times \frac{4 + 18 - 36}{4 - 30 + 36} \\ &= \frac{-140}{1} \times \frac{-14}{10} = (-14)^2 = 196 \end{aligned}$$

$$\text{Also } (a + 4 b)^2 = (2 + 12)^2 = (14)^2 = 196$$

Summary of steps.

1. Find the prime factors of each numerator and denominator of the fractions.

2. By division, remove all the factors that are common to any numerator and denominator.

3. Find the product of the remaining factors of the numerators for the numerator of the result, and the product of the remaining factors of the denominators for the denominator of the result.

Step (2) is frequently referred to as *canceling like factors*.

EXERCISES

Find the indicated products:

1. $\frac{3}{7} \times \frac{5}{4}$

8. $\frac{2}{3} \times 7\frac{1}{2}$

12. $\frac{ab}{cd} \times \frac{bd}{ac}$

2. $\frac{25}{36} \times \frac{42}{35}$

9. $\frac{2a}{b} \times \frac{b}{3c}$

13. $\frac{15 ab^2}{9 ax^2} \times \frac{7x}{70b}$

3. $\frac{18}{24} \times \frac{16}{27}$

10. $\frac{5x}{3y} \times \frac{9}{15x}$

14. $\frac{18 a^3 b^2}{14 x^3 y^2} \times \frac{35 x^2}{27 ab^3}$

4. $\frac{36}{42} \times \frac{28}{57}$

5. $\frac{45}{81} \times 3\frac{3}{5}$

11. $\frac{16 a^2}{xy} \times \frac{5x}{a}$

15. $\frac{25 ax}{32 mx^3} \times \frac{40}{125 a^3}$

6. $\frac{9}{36} \times 4\frac{2}{3}$

7. $\frac{32}{84} \times 5\frac{1}{4}$

Find the products:

16. $\frac{a+b}{5} \times \frac{10}{a^2-b^2}$

17. $\frac{x^2-9}{14} \times \frac{7}{x+3}$

18. $\frac{85}{x^2-16} \times \frac{x-4}{51}$

19. $\frac{3x^2}{x^2-64y^2} \times \frac{(x+8y)^2}{3x^2+24xy}$

20. $\frac{a-3b}{2x+1} \times \frac{4x^2-1}{a^2-9b^2}$

21. $\frac{4a+b}{p+5q} \times \frac{p^2-25q^2}{16a^2-b^2}$

22. $\frac{a^2+3a}{9xy} \times \frac{3axy}{a^3+3a^2}$

23. $\frac{a+b}{b-c} \times \frac{b^2-c^2}{a^2+ab}$

24. $\frac{a+3}{a-5} \times \frac{5a-25}{6+2a}$

25. $\frac{2ax+x^2}{c^2+bc} \times \frac{b^2-c^2}{x^2-4a^2}$

26. $\frac{a^2+a-6}{a^2-a-6} \times \frac{a^2+a-12}{a^2-a-12}$

27. $\frac{x^2-y^2}{a+b} \times \frac{a(a+b)}{x+y}$

28. $\frac{x^2-9}{(x-3)^2} \times \frac{x^2-8x+15}{x^2-2x-15}$

29. $\frac{a^4-b^4}{a^2-2ab+b^2} \times \frac{a-b}{a^2+ab}$

30. $\frac{a^2-a-6}{a^2-4} \times \frac{a+2}{a-3}$

31. $\frac{6a^2-15a+6}{6a^2+a-2} \times \frac{15a-3}{5a+10}$

32. $\frac{a^4-16}{625-x^4} \times \frac{x^2+25}{4+a^2}$

33. $\frac{2a^2-13a-7}{2a^2+7a+3} \times \frac{5a+15}{a^2-7a}$

34. $\frac{x^2-16}{3x^2-27} \times \frac{2x^2+2x-12}{x^2+x-20}$

35. $\frac{2a^2+ab-6b^2}{2a^2-5ab+3b^2} \times \frac{a^2-b^2}{a^2+4ab+4b^2}$

36. $\frac{2a^2+7ab+3b^2}{a^2-6ab+9b^2} \times \frac{3a-9b}{2a^2-ab}$

37. $\frac{6a^2-17ab-14b^2}{10a^2-35ab} \times \frac{25a^2}{3a^2-4ab-4b^2}$

38. $\frac{5a^2+15ab-50b^2}{3a^2+18ab+15b^2} \times \frac{6a^2-6b^2}{10a^2-30ab+20b^2}$

39. $\frac{a^2-b^2+2bc-c^2}{3a^2-3ab+3ac} \times \frac{6a^2}{c^2-bc-ac}$

Dividing Fractions

You will remember that to divide $\frac{9}{16}$ by $\frac{3}{4}$ you invert the divisor and proceed as in multiplication.

$$\text{Thus, } \frac{9}{16} \div \frac{3}{4} = \frac{9}{16} \times \frac{4}{3} = \frac{3}{4}$$

When a fraction is inverted, as when $\frac{3}{4}$ becomes $\frac{4}{3}$, the result is called the **reciprocal** of the original fraction. $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$ and $\frac{1}{5}$ is the reciprocal of 5.

EXAMPLE 1. Divide $\frac{a^2}{b^2}$ by $\frac{ay}{bx}$.

$$\text{SOLUTION. } \frac{a^2}{b^2} \div \frac{ay}{bx} = \frac{a^2}{b^2} \times \frac{bx}{ay} = \frac{ax}{by}$$

EXAMPLE 2. Divide $\frac{a^2 - b^2}{a^2 - 3ab + 2b^2}$ by $\frac{3}{5a - 10b}$.

SOLUTION.

$$\frac{a^2 - b^2}{a^2 - 3ab + 2b^2} \div \frac{3}{5a - 10b} = \frac{(a-b)(a+b)}{(a-b)(a-2b)} \times \frac{5(a-2b)}{3} = \frac{5(a+b)}{3}$$

Invert the fraction that is the divisor and proceed as in the multiplication of fractions. Check by substitution.

EXERCISES

Find the quotients:

1. $\frac{15}{27} \div \frac{30}{18}$

2. $\frac{25}{36} \div \frac{35}{18}$

3. $\frac{54}{72} \div \frac{9}{16}$

4. $\frac{55}{125} \div \frac{33}{75}$

5. $\frac{a}{b} \div \frac{x}{y}$

6. $\frac{5a}{b^2x} \div \frac{15}{4bx^2}$

7. $\frac{ab}{cd} \div \frac{bd}{ac}$

8. $\frac{a^2}{b^2} \div \frac{ax}{by}$

9. $\frac{2a^3x^2}{5b^2y^3} \div \frac{7b^3y}{5a^4x}$

10. $\frac{18a^2}{45x^3} \div \frac{4a}{5x}$

11. $\frac{abc}{bcx} \div \frac{a^2b}{cxy}$

12. $\frac{15x}{27} \div \frac{35y}{36}$

$$13. \frac{a^2 - b^2}{a^2 - ab} \div \frac{ab + b^2}{a}$$

$$15. \frac{9a^2 - b^2}{15ab} \div \frac{3ab - b^2}{b^2}$$

$$14. \frac{x^2 + 2xy}{xy + 2y^2} \div \frac{2x^2}{xy}$$

$$16. \frac{x^2 - y^2}{a^2 + ab} \div \frac{x + y}{a^2 - b^2}$$

$$17. \frac{25ax}{49by} \div \frac{5a^2 + 5a}{7ab - 14b}$$

$$18. \frac{75a^3b^2c}{36xy^2} \div \frac{5a^3b^2c^2 + 5a^2b^2c^2}{9ax^2y^2 - 9x^2y^2}$$

$$19. \frac{6a^2 - 11ab - 10b^2}{3a^2 + 5ab + 2b^2} \div (3a - 2b)$$

$$20. \frac{a^3 - 8a^2 + 16a}{a^2 + 2a} \div (a - 4)$$

$$21. \frac{x^2 - y^2}{x^2 + xy} \times \frac{x^3 + 3x^2y}{x^2 - xy} \div \frac{x^2 + 5xy + 6y^2}{x^2 + 2xy}$$

$$22. \frac{2x^2 + 5xy - 12y^2}{3x^2 + 11xy - 4y^2} \times \frac{12x^2 + 26xy - 10y^2}{2x^2 - xy - 3y^2} \div (2x + 5y)$$

$$23. \frac{a^2 - 11a + 30}{a^3 - 6a^2 + 9a} \times \frac{a^2 - 3a}{a^2 - 25} \div \frac{a^2 - 9}{a^2 + 2a - 15}$$

$$24. \frac{x^2 + 4x + 3}{x^2 - 8x + 7} \times \frac{x^2 - 2x - 35}{x^2 - 7x - 8} \div \frac{x^2 + 8x + 15}{x^2 - 9x + 8}$$

$$25. \frac{a^2 + a - 2}{a^2 + 2a - 3} \times \frac{a^2 + a - 6}{5a^2 + 25a} \div \frac{a^2 - 4}{4ab + 20b}$$

$$26. \frac{2a^2 - ab - b^2}{a^2 + 3ab + 2b^2} \times \frac{a^2 - 2ab + b^2}{6a^2 + 7ab + 2b^2} \div \frac{a^2 - 2ab + b^2}{3a^2 - ab - 2b^2}$$

$$27. \frac{a^2 - (b - c)^2}{b^2 - (a + c)^2} \div \frac{a + b - c}{a - b + c}$$

$$28. \frac{(a + b)^2 - 4}{4 - (b - a)^2} \div \frac{a + b - 2}{a - b + 2}$$

$$29. \frac{2a^2 - ax - 4ab + 2bx}{2a^2 + 5ax - 3x^2} \div \frac{2a^2 + a - 4ab - 2b}{2a^2 - a + 6ax - 3x}$$

$$30. \frac{ab + 3a + 2b + 6}{ab - 2b + 3a - 6} \div \frac{a^2 + 6a - 16}{a^2 + a - 6}$$

Reducing Fractions to Integral Form or to Mixed Expressions

In arithmetic, as you know, a fraction is changed to a mixed number by dividing the numerator by the denominator and expressing the quotient with the remainder:

$$\text{Thus, } \frac{15}{4} = 15 \div 4 = 3 + \frac{3}{4} = 3\frac{3}{4}$$

Reduction of fractions in algebra is done in the same way.

EXAMPLE. Reduce the fraction $\frac{2a+b}{a}$ to a mixed expression.

SOLUTION. Divide each term of $2a + b$ by a .

$$\frac{2a+b}{a} = \frac{2a}{a} + \frac{b}{a} = 2 + \frac{b}{a}$$

Check by substitution.

Note that the sign $+$ must be written between 2 and $\frac{b}{a}$ since $2\frac{b}{a}$ means $\frac{2b}{a}$, which is not the same as $2 + \frac{b}{a}$.

Reduce the fraction to lowest terms and then divide the numerator by the denominator.

EXERCISES

Express in integral form or as mixed expressions :

1. $\frac{15ab}{5a}$

6. $\frac{a^2 + 2ab}{a^2}$

11. $\frac{a^2 + 2ab + b^2}{a}$

2. $\frac{20ab^2}{4ab}$

7. $\frac{a^2 - 2ab}{ab}$

12. $\frac{x^3 + 2x^2 + 3x}{x^2}$

3. $\frac{ax + b}{ax}$

8. $\frac{x^2 + 3xy}{x + 3y}$

13. $\frac{x^2y^2 + 3xy + 2}{xy}$

4. $\frac{a^2 + 2ab}{a}$

9. $\frac{x^2 + 2x + 1}{x}$

14. $\frac{10x^2 + 5x - 15}{5x}$

5. $\frac{8a^2 - 4b^2}{4a}$

10. $\frac{x^2 + 3x + 4}{x + 3}$

15. $\frac{a^2 + 2ab + b^2}{a + 2b}$

16. $\frac{3a^2 + 9ab - 12b^2}{3a^2 + 12ab}$

18. $\frac{a^3 + 3a^2b - 10ab^2}{a^2 + 3ab}$

17. $\frac{a^2 - ab - 6b^2}{a^2 + 2ab}$

19. $\frac{a^3 + a^2x + a}{a^2 + ax}$

Finding the lowest common multiple.

Before fractions can be added, they must have the same denominators. You can add $\frac{1}{5}$ and $\frac{3}{5}$ and get $\frac{4}{5}$; but before you can add $\frac{1}{3}$ and $\frac{1}{5}$ you must reduce them to fractions having the same denominator.

Thus, $\frac{1}{3} = \frac{5}{15}$ and $\frac{1}{5} = \frac{3}{15}$; then $\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$

A **multiple** of a number is any number that exactly contains the given number.

Thus, 10, 15, 20, etc. are multiples of 5; also $7a$, a^2x , $a(x+y)$, etc. are multiples of a .

A **common multiple** of two or more numbers is a number that exactly contains each of the given numbers.

Thus, 30 is a common multiple of 2, 3, and 5; also abc is a common multiple of a , b , and c .

The **lowest common multiple (L. C. M.)** of two or more numbers is their common multiple that contains the fewest factors.

Thus, while 24 is a common multiple of 2, 3, and 4, their lowest common multiple is 12. The L. C. M. of $a+b$ and $a-b$ is $a^2 - b^2$.

EXAMPLE 1. Find the L. C. M. of 15, 18, and 20.

SOLUTION. Find the prime factors of each number.

$$15 = 5 \times 3, \quad 18 = 2 \times 3^2, \quad \text{and} \quad 20 = 5 \times 2^2$$

The L. C. M. must contain 15, or 5×3 . It must also contain 18 and 20, hence it must be a multiple of 2×3^2 and of 5×2^2 .

Hence the L. C. M. of 15, 18, and 20 = $5 \times 3^2 \times 2^2$, or 180.

CHECK. $180 \div 15 = 12$; $180 \div 18 = 10$; $180 \div 20 = 9$.

The L. C. M. of two or more numbers is the product of each different factor of the numbers taken the greatest number of times it occurs in any one of the numbers.

EXAMPLE 2. Find the L. C. M. of $6ab$, $8b^3c^2$, and $9a^2x$.

SOLUTION. The L. C. M. of 6, 8, and 9 is 72. The product of each of the literal factors taken the greatest number of times it is contained in any expression is $a^2b^3c^2x$.

Hence the L. C. M. of $6ab$, $8b^3c^2$, and $9a^2x$ is $72a^2b^3c^2x$.

EXAMPLE 3. Find the L. C. M. of $4a^2 + 8ab + 4b^2$ and $3a^2 - 3b^2$.

SOLUTION. $4a^2 + 8ab + 4b^2 = 4(a + b)^2$

$$3a^2 - 3b^2 = 3(a + b)(a - b)$$

Hence the L. C. M. of the expressions is $12(a + b)^2(a - b)$.

Check by dividing $12(a + b)^2(a - b)$ by each of the expressions.

Summary of steps.

1. Find the prime factors of each expression.
2. Find the product of each of the different factors of the expressions, giving each factor the largest exponent it has in any of the expressions.

EXERCISES

Find the L. C. M. of :

- | | |
|---|-------------------------------------|
| 1. 15 and 12 | 9. $12a^2x$, $8ay^3$, and $15x^2$ |
| 2. 8 and 20 | 10. $a + b$ and $a - b$ |
| 3. 12, 18, and 20 | 11. $x - 2y$ and $x + 3y$ |
| 4. 15, 20, and 24 | 12. $ab + ac$ and $b^2 + bc$ |
| 5. $15ax$ and $6bx$ | 13. $a^2 + ab$ and $ab + b^2$ |
| 6. $12a^2x$ and $8bx^3$ | 14. $3x - 3y$ and $x^2 - y^2$ |
| 7. $9a^2x$ and $8ab^3x^4$ | 15. $x^2 + y^2$ and $(x + y)^2$ |
| 8. $18r^2s$, $6rs^2$, and $4r^3x$ | 16. $a^2 - b^2$ and $(a - b)^2$ |
| 17. $a + b$, $a - b$, and $3a^2 - 3b^2$ | |
| 18. $4a^3 - 9a$, $6a^2 - 9a$, and $4a^3 - 12a^2 + 9a$ | |
| 19. $2a^4 - 2a^2$, $2a^3 - 4a^2 + 2a$, and $4a^2 - 4a$ | |
| 20. $x^2 + 5x + 6$, $x^2 + 4x + 4$, and $x^2 + 6x + 9$ | |
| 21. $x^3 - 16x$, $x^2 - x - 12$, and $x^3 + 7x^2 + 12x$ | |
| 22. $x^2 - y^2$, $x^2 + xy$, and $x^2y + xy^2$ | |

Find the L. C. M. of the denominators of the fractions :

23. $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, and $\frac{2}{3}$

24. $\frac{7}{ax}$, $\frac{4}{3a^2x}$, and $\frac{5}{7ax^2}$

25. $\frac{a}{x + y}$, $\frac{b}{x - y}$, and $\frac{c}{x^2 - y^2}$

Reducing Fractions to Equivalent Fractions

Equivalent fractions are fractions that have the same value when reduced to lowest terms. They differ only in form.

Thus, $\frac{6}{8}$, $\frac{15}{20}$, and $\frac{21}{28}$ are equivalent fractions since each equals $\frac{3}{4}$; also $\frac{a+b}{a-b}$ is equivalent to $\frac{a^2+ab}{a^2-ab}$.

Fractions that are to be combined by addition or subtraction should first be reduced to equivalent fractions that have for their common denominator the L. C. M. of their denominators. This denominator is called the **lowest common denominator (L. C. D.)** of the fractions. (See page 74.)

EXAMPLE 1. Reduce $\frac{5}{6}$ and $\frac{3}{8}$ to equivalent fractions having their L. C. D.

SOLUTION. Their L. C. D. is the L. C. M. of 6 and 8, which is 24. $24 = 6 \times 4$; hence to make the denominator of $\frac{5}{6}$ equal 24 it is necessary to multiply it by 4. Since the new fraction is to be equivalent to $\frac{5}{6}$, both terms must be multiplied by 4.

$$\text{Hence } \frac{5}{6} = \frac{4 \times 5}{4 \times 6} = \frac{20}{24}. \quad \text{Similarly, } \frac{3}{8} = \frac{3 \times 3}{3 \times 8} = \frac{9}{24}$$

PRINCIPLE. Both terms of a fraction may be multiplied by the same factors without changing the value of the fraction.

EXAMPLE 2. Reduce $\frac{x}{a^2+ax}$ and $\frac{a}{a^2-x^2}$ to equivalent fractions having their L. C. D.

SOLUTION. The L. C. D. is $a(a+x)(a-x)$. Divide $a(a+x)(a-x)$ by each denominator; then multiply each term of the fraction by the corresponding quotients.

$$\begin{aligned} \frac{x}{a(a+x)} &= \frac{(a-x)x}{a(a-x)(a+x)} = \frac{ax-x^2}{a(a-x)(a+x)} \\ \frac{a}{(a-x)(a+x)} &= \frac{a \cdot a}{a(a-x)(a+x)} = \frac{a^2}{a(a-x)(a+x)} \end{aligned}$$

Summary of steps.

1. Find the L. C. D. of the denominators of the fractions.
2. Multiply the numerator and the denominator of each fraction by the factor that will make the denominator of the fraction equal to the L. C. D. of the fractions.

EXERCISES

Reduce to equivalent fractions having their lowest common denominator :

1. $\frac{a}{4}, \frac{a}{6}$

2. $\frac{ax}{8}, \frac{ax}{12}$

3. $\frac{a}{2}, \frac{a}{3}, \frac{a}{4}$

4. $\frac{x}{4}, \frac{y}{3}, \frac{xy}{6}$

5. $\frac{a}{12}, \frac{ab}{15}, \frac{b}{18}$

6. $\frac{a}{9}, \frac{ax}{12}, \frac{ab}{15}$

7. $\frac{5}{a}, \frac{3}{b}$

8. $\frac{x-1}{3}, \frac{x+2}{4}$

9. $\frac{n+1}{2}, \frac{n-1}{3}$

10. $\frac{2n-1}{5}, \frac{n+3}{4}$

11. $\frac{5}{a}, \frac{1}{b}, \frac{-3}{ab}$

12. $\frac{4}{x}, \frac{7}{y}, \frac{-2}{xy}$

13. $\frac{a+b}{a}, \frac{a-b}{b}, 2$

14. $\frac{a}{a+b}, \frac{b}{a-b}$

15. $\frac{3}{a-3}, \frac{5}{a+3}$

16. $\frac{5}{a+b}, \frac{3}{a^2-b^2}$

17. $\frac{2}{x^2+xy}, \frac{-3}{xy-y^2}$

18. $\frac{b}{a^2+ab}, \frac{a}{ab+b^2}$

19. $\frac{5}{x+3}, \frac{2}{x-2}$

20. $\frac{5a}{x^2+3x}, \frac{7b}{2x+6}$

21. $\frac{x}{x^2+2x}, \frac{y}{xy+2y}$

22. $\frac{a}{a^2+3ab}, \frac{8}{a^2-9b^2}$

23. $\frac{a+b}{a-b}, \frac{a-b}{a+b}$

24. $\frac{5}{x-7}, \frac{8}{x+3}$

25. $\frac{3}{2x+6}, \frac{7}{3x+9}$

26. $\frac{5x}{x+8y}, \frac{7x^2}{x^2-64y^2}$

27. $\frac{2ax}{a^2+3a}, \frac{2}{a-7}, \frac{-7}{a^2-4a-21}$

28. $\frac{3}{x^2+3x-10}, \frac{-4}{x^2-25}, \frac{1}{x^2-7x+10}$

Adding and Subtracting Fractions

EXAMPLE 1. Add $\frac{a+b}{12}$ and $\frac{a-2b}{15}$.

SOLUTION. Proceed as in arithmetic. Change the fractions to equivalent fractions having their L. C. D. Then find the sum of the numerators for the numerator of the result. The L. C. D. is the denominator of the result.

The L. C. D. of 12 and 15 is 60.

$$\begin{aligned}\frac{a+b}{12} + \frac{a-2b}{15} &= \frac{5(a+b)}{5 \times 12} + \frac{4(a-2b)}{4 \times 15} = \frac{5a + 5b + 4a - 8b}{60} \\ &= \frac{9a - 3b}{60} = \frac{3a - b}{20}\end{aligned}$$

Check by substitution.

NOTE. Reduce the result to lowest terms.

EXAMPLE 2. Add $\frac{a+b}{2b} - \frac{a-b}{3a}$.

SOLUTION. The L. C. D. of $2b$ and $3a$ is $6ab$.

$$\text{Then } \frac{a+b}{2b} - \frac{a-b}{3a} = \frac{3a(a+b)}{3a \times 2b} - \frac{2b(a-b)}{2b \times 3a} \quad (1)$$

$$= \frac{3a(a+b) - 2b(a-b)}{6ab} \quad (2)$$

$$= \frac{3a^2 + 3ab - 2ab + 2b^2}{6ab} \quad (3)$$

$$= \frac{3a^2 + ab + 2b^2}{6ab} \quad (4)$$

CHECK. Let $a = 2$, $b = 3$.

$$\text{Then } \frac{a+b}{2b} - \frac{a-b}{3a} = \frac{2+3}{6} - \frac{2-3}{6} = \frac{5}{6} - \frac{-1}{6} = \frac{5+1}{6} = 1$$

$$\text{Also } \frac{3a^2 + ab + 2b^2}{6ab} = \frac{12 + 6 + 18}{36} = \frac{36}{36} = 1$$

NOTE. In (1) the fractions are written as equivalent fractions having their L. C. D.

In (2) the numerators of the fractions in (1) are written as one numerator over one denominator. The minus sign before the second fraction is retained before the second numerator. To avoid mistakes in signs, the parentheses in the numerator of (2) are retained.

In (3) the parentheses are removed.

In (4) the numerator is simplified.

Summary of steps.

1. Change the fractions, if necessary, to equivalent fractions having their lowest common denominator.
2. For the numerator of the result write the numerators of the resulting fractions, preceded by the sign of the fraction. For the denominator of the result write the L. C. D.
3. Simplify the numerator.
4. Reduce the fraction to its lowest terms.

EXERCISES

Combine the fractions as indicated :

1. $\frac{5}{12} + \frac{3}{8} + \frac{2}{3}$

2. $\frac{1}{4} + \frac{2}{3} + \frac{5}{6}$

3. $\frac{a}{15} + \frac{a}{12} + \frac{2a}{10}$

4. $\frac{2a}{9} + \frac{3a}{8} - \frac{a}{12}$

5. $\frac{x}{15} - \frac{2x}{9} + \frac{5x}{6}$

6. $\frac{a}{24} + \frac{3a}{20} - \frac{7a}{15}$

7. $\frac{3x}{4} + \frac{2x}{9} + \frac{x}{6}$

8. $\frac{5x}{18} - \frac{4x}{15} + \frac{x}{9}$

9. $\frac{7a}{16} + \frac{5a}{12} - \frac{7a}{24}$

10. $\frac{5b}{14} + \frac{2b}{7} - \frac{3b}{2}$

21. $\frac{a}{b} + \frac{b}{a}$

22. $\frac{b}{a} - \frac{a}{b}$

23. $\frac{a}{b} + \frac{b}{c}$

24. $\frac{a}{c} - \frac{b}{d}$

11. $\frac{a+b}{2} + \frac{a-b}{3}$

12. $\frac{x-y}{5} - \frac{x+y}{2}$

13. $\frac{2a-b}{16} + \frac{b-a}{12}$

14. $\frac{x+3y}{18} - \frac{2x-y}{24}$

15. $\frac{3x-2y}{12} + \frac{3x+5y}{21}$

16. $\frac{2a+b}{6} - \frac{a-2b}{8}$

17. $\frac{3a-2b}{8} + \frac{2a+b}{20}$

18. $\frac{5x-y}{18} - \frac{2y+x}{4}$

19. $\frac{2x-y}{36} + \frac{3x-5y}{45}$

20. $\frac{5x+3y}{24} - \frac{2x+7y}{36}$

25. $\frac{m-n}{m} - \frac{m+n}{n}$

26. $\frac{a+b}{a} + \frac{a-b}{b}$

Combine the fractions as indicated :

$$27. \frac{2x-1}{3x} + \frac{x-3}{6x}$$

$$28. \frac{x-2y}{2y} - \frac{2x+3y}{8x}$$

$$29. \frac{2x+7y}{x^2y} + \frac{3x-5y}{xy^2}$$

$$30. \frac{a^2+ab}{4a^2} - \frac{ab-b^2}{6b^2}$$

$$33. \frac{a^2-1}{2a^2-2a} - \frac{a^2-1}{3a^2}$$

$$34. \frac{a^2-2ab+b^2}{6ab} + 1$$

$$35. \frac{a^2+b^2}{ab} + 2$$

$$36. \frac{a}{a+b} - \frac{b}{a-b}$$

$$37. \frac{2}{2x-1} - \frac{3}{x-2}$$

$$38. \frac{2a}{a-2b} + \frac{3a}{2a-b}$$

$$39. \frac{5a}{5x-3y} + \frac{2a}{x-2y}$$

HINT. First reduce the fractions to lowest terms.

$$31. \frac{a^2+2ab+b^2}{12a} - \frac{ab}{3b}$$

$$32. \frac{x^2+xy}{(x+y)^2} + \frac{y}{x-y}$$

$$40. \text{ Find } \frac{a+2}{2a^2+a-1} - \frac{a-3}{4a^2-1}$$

SOLUTION.

$$\begin{aligned} \frac{a+2}{2a^2+a-1} - \frac{a-3}{4a^2-1} &= \frac{a+2}{(2a-1)(a+1)} - \frac{a-3}{(2a-1)(2a+1)} \\ &= \frac{(a+2)(2a+1) - (a-3)(a+1)}{(2a-1)(a+1)(2a+1)} \\ &= \frac{2a^2+5a+2 - a^2+2a+3}{(2a-1)(a+1)(2a+1)} \\ &= \frac{a^2+7a+5}{(2a-1)(a+1)(2a+1)} \end{aligned}$$

Combine the fractions as indicated :

$$41. \frac{a}{ab+b^2} + \frac{b}{a^2+ab}$$

$$44. \frac{2}{3x+6} + \frac{5x}{x^2-4}$$

$$42. \frac{x+2}{x^2+5x+6} + \frac{x+3}{x^2+7x+6}$$

$$45. \frac{3x+1}{x^2-3x+2} - \frac{x-3}{x-2}$$

$$43. \frac{s+2t}{s+3t} - \frac{2s+5t}{2s+7t}$$

$$46. \frac{2x-5}{2x^2+x-6} + \frac{5}{3x+6}$$

$$47. \frac{2a+1}{a^2-5a+6} - \frac{a}{a^2-6a+8} - \frac{a-5}{a^2-7a+12}$$

The Sign of a Fraction

The sign of a fraction is written before the fraction, thus:

$-\frac{a}{b}$. This means that the fraction $\frac{a}{b}$ is a negative quantity.

Besides the sign of the fraction as a whole, there is the sign of the numerator and the sign of the denominator.

Thus, $-\frac{a}{b}$ means $-\frac{+a}{+b}$. Each of the signs may be $+$ or $-$. If no sign is stated, the sign $+$ is understood.

PRINCIPLE. *Both terms of a fraction may be multiplied by the same number without changing the value of the fraction.* (See page 191.)

According to this principle and the law of signs in division, the following fractions are equal:

$$\begin{array}{ccc} (1) & (2) & (3) \\ -\frac{a}{+b} & = \frac{+a}{-b} & = -\frac{+a}{+b} \end{array}$$

(2) is obtained from (1) by multiplying both terms by -1 :

$$\frac{-1(-a)}{-1(b)} = \frac{+a}{-b}$$

(3) follows from (1) and (2), according to the law of signs in division. (See page 67.)

Note that there are three signs to be considered: *the sign of the fraction, the sign of the numerator, and the sign of the denominator.*

From the above it follows that:

The sign of one of the terms of a fraction may be changed, provided the sign of the other term is changed or the sign before the fraction is changed.

That is, the value of the fraction is not altered if two of the three signs of the fraction are changed.

$$\text{Show that } +\frac{-2}{-3} = -\frac{-2}{+3} = +\frac{+2}{+3}.$$

ORAL EXERCISES

Explain how the indicated changes are made and show that the relations are true:

$$1. \frac{-3}{4} = -\frac{3}{4}$$

$$2. \frac{2}{-3} = -\frac{2}{3}$$

$$3. \frac{-a}{b} = -\frac{a}{b}$$

$$4. \frac{-a}{-b} = +\frac{a}{b}$$

$$5. \frac{1}{b-a} = -\frac{1}{a-b}$$

$$6. -\frac{x}{y-x} = +\frac{x}{x-y}$$

$$7. -\frac{-a}{b-a} = -\frac{a}{a-b}$$

$$8. \frac{3-2a}{5-a} = \frac{2a-3}{a-5}$$

$$9. \frac{3+a}{2-a} = -\frac{a+3}{a-2}$$

$$10. \frac{x^2-2}{5+4x-x^2} = -\frac{x^2-2}{x^2-4x-5}$$

$$11. \text{ Change } \frac{b}{4-2b} \text{ to a fraction with denominator } 2b-4.$$

$$12. \frac{6}{3-a} = \frac{?}{a-3}$$

$$14. \frac{b}{4-2b} = \frac{?}{2b-4}$$

$$13. \frac{3ax-x^2}{x-a} = \frac{?}{a-x}$$

$$15. \frac{-2a}{4-a^2} = \frac{?}{a^2-4}$$

$$16. \text{ In the fraction } \frac{(a-2b)(a-b)(2a-b)}{2a-3b}, \text{ what is the}$$

effect on the sign of the fraction of changing the sign of one of the factors? the sign of two of the factors? the sign of three of the factors? the sign of four of the factors? Explain each case.

From a consideration of the changes of sign noticed in ex. 16 it follows that:

The sign of a fraction is not changed if the signs of an even number of factors are changed.

The sign of a fraction is changed if the signs of an odd number of factors are changed.

EXERCISES

1. Combine the fractions $\frac{2}{x+2}$ and $\frac{3}{2-x}$.

SOLUTION. Change the second fraction so that the denominators are arranged in the same order.

$$\begin{aligned}\frac{2}{x+2} + \frac{3}{2-x} &= \frac{2}{x+2} - \frac{3}{x-2} = \frac{2(x-2) - 3(x+2)}{(x+2)(x-2)} \\ &= \frac{2x-4-3x-6}{(x+2)(x-2)} = \frac{-x-10}{x^2-4} = \frac{x+10}{4-x^2}\end{aligned}$$

Combine the following fractions:

2. $\frac{a}{a-b} - \frac{b}{b-a}$
3. $\frac{x}{x+y} + \frac{y}{y-x}$
4. $\frac{5x}{x^2-y^2} - \frac{3}{y-x}$
5. $\frac{1-x}{1+x} + \frac{1+x}{x-1}$
6. $\frac{x+3}{x+2} - \frac{2+x}{5-x}$
7. $\frac{x-2}{x+3} + \frac{2x-1}{2-x}$
8. $\frac{a-b}{a+b} - \frac{a+b}{b-a}$
9. $\frac{a-1}{a+1} - \frac{a+1}{1-a}$
10. $\frac{x}{x+y} + \frac{x}{y-x}$
11. $\frac{1}{a+2} - \frac{1}{4-a^2}$
12. $\frac{1}{a+3} + \frac{3}{9-a^2}$
13. $\frac{1}{x-4} + \frac{x}{16-x^2}$
14. $\frac{x-1}{x+2} + \frac{x^2-3}{4-x^2} + \frac{x}{2+x}$
15. $\frac{5x+2}{x^2-4} + \frac{2}{x-2} - \frac{3}{2-x}$
16. $\frac{3ab}{a^2-4b^2} + \frac{b+a}{2b-a} - \frac{b-a}{2b+a}$
17. $\frac{x+y}{x-y} + \frac{4xy}{y^2-x^2} - \frac{x-y}{x+y}$

Exercises on Combined Operations

1. $\left(1 + \frac{x}{y}\right) \div \left(\frac{x}{y} - 1\right) = ?$

SOLUTION. $\left(1 + \frac{x}{y}\right) = \frac{y+x}{y} = \frac{x+y}{y}; \quad \left(\frac{x}{y} - 1\right) = \frac{x-y}{y}$

$$\left(1 + \frac{x}{y}\right) \div \left(\frac{x}{y} - 1\right) = \frac{x+y}{y} \times \frac{y}{x-y} = \frac{x+y}{x-y}$$

Perform the indicated operations :

$$2. \left(a - \frac{a}{3}\right) \div \left(a + \frac{a}{3}\right) \qquad 7. \left(\frac{1}{a} - \frac{1}{b}\right) \div \left(\frac{1}{a} + \frac{1}{b}\right)$$

$$3. \left(1 + \frac{x}{y}\right) \div \left(1 - \frac{y^2}{x^2}\right) \qquad 8. \left(\frac{x}{a} + \frac{y}{b}\right) \div \left(\frac{a}{x} + \frac{b}{y}\right)$$

$$4. \left(1 - \frac{y}{x}\right) \div \left(x - \frac{y^2}{x}\right) \qquad 9. (x + y)\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$5. \left(a - \frac{b}{2}\right) \div \left(\frac{a}{2} + b\right) \qquad 10. \left(a + 1 + \frac{1}{b}\right)^2$$

$$6. \left(\frac{a}{c} + \frac{c}{a}\right)\left(\frac{c}{a} - \frac{a}{c}\right) \qquad 11. (a^2 - 2a)\left(1 + \frac{2}{a}\right)$$

$$12. \left(4 - \frac{3}{a+1}\right) \div \left(6 + \frac{5}{a^2-1}\right)$$

$$13. \left(2x + \frac{x}{x-2}\right) \div \left(2x - \frac{x}{x-2}\right)$$

$$14. \left(b + \frac{ab}{b-a}\right)\left(b - \frac{ab}{a+b}\right)\left(\frac{b^2-a^2}{b^2+a^2}\right)$$

$$15. \left(x - \frac{y^2}{x}\right)\left(\frac{x}{y} + \frac{y}{x}\right)\left(\frac{xy}{x^4-y^4}\right)$$

$$16. \left(3x - 5 - \frac{2}{x}\right)\left(3x + 5 - \frac{2}{x}\right) \div \left(x - \frac{4}{x}\right)$$

$$17. \left(\frac{x+y}{y} + \frac{y}{x+y}\right) \div \left(\frac{x+y}{xy}\right)$$

$$18. \left(2 - \frac{a+b}{a-b}\right) \div \left(2 + \frac{a+b}{a-b}\right)$$

$$19. \left(\frac{2}{x} - \frac{1}{a+x} + \frac{1}{a-x}\right) \div \left(\frac{a+x}{a-x} - \frac{a-x}{a+x}\right)$$

$$20. \left(\frac{x-2}{x-3} - \frac{x+3}{x+4}\right) \div \left(\frac{1}{x+1} + \frac{7}{x-3}\right)$$

$$21. \left(\frac{1}{x+1} + \frac{1}{x-1}\right) \div \left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

$$22. \left(\frac{a+3b}{a-3b} + \frac{a-3b}{a+3b}\right) \div \left(\frac{a-3b}{a+3b} - \frac{a+3b}{a-3b}\right)$$

Complex Fractions

A fraction that has a fraction in either its numerator or its denominator or in both is called a **complex fraction**.

Thus, $\frac{\frac{1}{2}}{\frac{3}{4}}$, $\frac{\frac{2}{3}}{\frac{3}{4}}$, and $\frac{\frac{b}{c}}{\frac{d}{e}}$ are complex fractions.

EXAMPLE. Simplify the fraction $\frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x} - 2}$.

SOLUTION. Consider the fraction as an exercise in division.

$$\begin{aligned} \left(\frac{x}{y} - \frac{y}{x}\right) \div \left(\frac{x}{y} + \frac{y}{x} - 2\right) &= \frac{x^2 - y^2}{xy} \div \frac{x^2 + y^2 - 2xy}{xy} \\ &= \frac{(x+y)(x-y)}{xy} \times \frac{xy}{(x-y)(x-y)} = \frac{x+y}{x-y} \end{aligned}$$

EXERCISES

1. Simplify the fraction in the example above by multiplying both terms by xy , the L. C. M. of the denominators x and y .
2. Formulate a rule for simplifying a complex fraction.

Simplify the following fractions:

3. $\frac{\frac{2a}{3}}{\frac{3b}{4}}$

6. $\frac{\frac{2}{a}}{5 + \frac{1}{a}}$

9. $\frac{1 + \frac{a}{b}}{1 - \frac{a}{b}}$

12. $\frac{\frac{b+a}{b-a} - 2}{\frac{b}{a} - \frac{b}{b-a}}$

4. $\frac{\frac{2}{3a}}{\frac{6a^2}{10}}$

7. $\frac{2a - \frac{2}{3}}{2a + \frac{2}{3}}$

10. $\frac{\frac{a}{b} - \frac{b}{a}}{\frac{1}{a} + \frac{1}{b}}$

13. $\frac{1 - \frac{x}{x-y}}{\frac{x}{x-y}}$

5. $\frac{\frac{3x}{4y}}{\frac{16y^2}{12x^2}}$

8. $\frac{x + \frac{y}{z}}{x - \frac{y}{z}}$

11. $\frac{\frac{a}{2} - \frac{b}{3}}{\frac{a}{3} + \frac{b}{2}}$

14. $\frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}}$

REVIEW EXERCISES

1. A fraction is ——. The terms of a fraction are —.

2. If 3 is added to both terms of $\frac{9}{12}$, is the value of the fraction increased or decreased?

3. If 3 is subtracted from both terms of $\frac{9}{12}$, the value of the fraction is —.

4. If x increases, what change takes place in the value of $\frac{x+7}{8}$? of $\frac{5}{x+3}$? If x decreases, what change occurs?

5. If x is positive and decreases, what change takes place in the value of $\frac{15-x}{25}$? What change occurs if x is negative?

6. If x is negative and increases in absolute value, what change takes place in the value of $(5-x)^2 - x$?

7. If $y = \frac{2x^2 - 3}{x}$, the value of y is a function of —.

8. In ex. 7, if x increases, then y —.

9. If $y = kx$, y is said to vary —.

10. If $y = \frac{k}{x}$, y is said to vary —.

11. If $y = kx^2$, y is said to vary —.

12. If the signs of an odd number of factors of a fraction are changed, then —; if the signs of an even number of factors are changed, then —.

13. Is the value of the fraction $\frac{(a-b)(c-b)}{(a-c)(c-a)}$ the same as the value of $\frac{(a-b)(b-c)}{(a-c)(c-a)}$ or $\frac{(a-b)(b-c)}{(a-c)(a-c)}$? In each case explain the changes made and state the principle involved.

14. A fraction is in its lowest terms if —.

15. The principle used in reducing a fraction to lower terms is —.

16. The H. C. F. of two or more expressions is —.

17. The L. C. D. of two or more fractions is —.

Reduce to lowest terms:

$$18. \frac{x^2 + 5x - 24}{15 + x - 2x^2}$$

$$19. \frac{18x^2 + 51x - 42}{6x^2 + 29x + 28}$$

$$20. \frac{8x^2 - 6x - 9}{6x^3 - 17x^2 + 12x}$$

$$21. \frac{6x^2 - 11x - 10}{6x^2 - 17x + 5}$$

$$22. \frac{x^3 - 3x^2 + 2x - 6}{x^2 + 2x - 15}$$

$$23. \frac{(x + y)^2 - 4z^2}{x^2 - (y + 2z)^2}$$

$$24. \frac{x^2 - y^2 - z^2 + 2yz}{x^2 + y^2 - z^2 + 2xy}$$

$$25. \frac{a^2 + 4b^2 - 4c^2 - 4ab}{a^2 - 4b^2 - 4c^2 - 8bc}$$

Perform the indicated operations:

$$26. \frac{2a - b}{3a + 3b} - \frac{a + b}{a - b} + \frac{a - b}{a + b}$$

$$27. \frac{3}{1 - 2x} - \frac{7}{1 + 2x} + \frac{4 - 20x}{1 - 4x^2}$$

$$28. \frac{x}{1 - x} - \frac{3x}{x + 1} + \frac{5x}{x^2 - 1}$$

$$29. \frac{x}{x - 2} - \frac{x - 2}{x + 2} + \frac{3}{4 - x^2}$$

$$30. \frac{1}{x - 2} - \frac{1}{(1 - x)^2} - \frac{1}{x - 1}$$

$$31. \frac{a + x}{2a - 2x} + \frac{x - a}{4x + 4a} + \frac{a^2 - 8ax - 5x^2}{8x^2 - 8a^2}$$

$$32. \frac{x + y}{x - y} - \frac{x^2 + y^2}{y^2 - x^2} + \frac{y - x}{x + y}$$

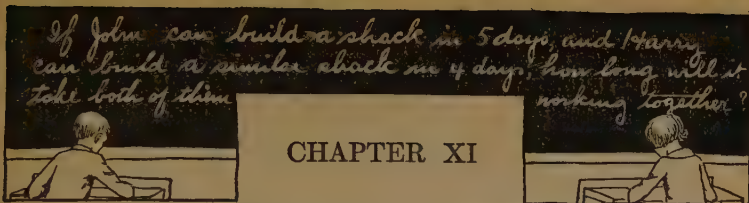
$$33. \frac{a}{ab + b^2} - \frac{b}{ab - a^2} + \frac{a}{a^2 - b^2}$$

$$34. \left(1 - \frac{t}{s + t}\right) \left(\frac{s + t}{s} - 1\right)$$

$$35. \left(\frac{5}{x^2 - 4} + \frac{2}{2 - x}\right) \div \left(2 + \frac{3}{x - 2}\right)$$

$$36. \frac{x^3 + x^2 - x - 1}{x^2 + 2x + 1} \div \frac{x^2 + x - 2}{5x + 10}$$

$$37. \text{ Put in fractional form } x - a + y + \frac{a^2 - ay + y^2}{x + a}.$$



CHAPTER XI

FRACTIONAL EQUATIONS

Many equations involve more difficult algebraic fractions than those which you have had up to this point. (See pages 74 and 100.) It is the purpose of this chapter to show the methods employed in solving such equations.

Clearing an equation of fractions.

EXAMPLE. Solve $\frac{5(2x - 1)}{3} - \frac{3x - 1}{5} = \frac{4(7x + 1)}{15}$.

SOLUTION. $\frac{5(2x - 1)}{3} - \frac{3x - 1}{5} = \frac{4(7x + 1)}{15}$ (1)

Multiply by 15: $25(2x - 1) - 3(3x - 1) = 4(7x + 1)$ (2)

Remove parentheses: $50x - 25 - 9x + 3 = 28x + 4$ (3)

Solve: $x = 2$ (4)

CHECK. $\frac{5(4 - 1)}{3} - \frac{(6 - 1)}{5} = \frac{4(14 + 1)}{15}$; $5 - 1 = 4$; $4 = 4$

Note that 15 is the L. C. D. of the denominators. The negative sign before the second fraction in (1) must be retained in (2) and a parenthesis must be placed around $(3x - 1)$ to show that the entire numerator is to be considered as one quantity. The parenthesis takes the place of the bar that indicates division.

The process of freeing an equation of fractions is called **clearing the equation of fractions**. Before clearing an equation of fractions, reduce all fractions to their lowest terms.

Summary of steps.

1. Reduce all fractions to their lowest terms.
2. Multiply both sides of the equation by the L. C. D. of the denominators of the fractions.
3. Simplify the resulting equation and solve it.
4. Check the result.

EXERCISES

Solve the following equations. Check ex. 1 to 10 and 15 to 22:

1. $\frac{x+1}{4} + \frac{x+3}{2} = 4$

12. $\frac{1}{3y} - \frac{1}{2y} = 1$

2. $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 7$

13. $\frac{2}{y} - \frac{4}{3y} = \frac{2}{15}$

3. $\frac{x}{3} - \frac{4x}{5} = \frac{7}{3}$

14. $\frac{7}{8y} - \frac{5}{6y} = \frac{1}{12}$

4. $\frac{x+3}{4} - \frac{2x-3}{7} = 1$

15. $\frac{1}{x} + \frac{2}{x} = 3 - \frac{3}{x}$

5. $\frac{3x-2}{3} - \frac{4x+1}{27} = \frac{x}{2}$

16. $\frac{5x+1}{3} - \frac{3-x}{5x} = \frac{25x^2+7}{15x}$

6. $\frac{5x-1}{6} + \frac{3x+1}{4} = 8$

17. $\frac{28}{15} - \frac{2x-1}{3x} = \frac{4x+1}{5x}$

7. $\frac{2x-3}{3} + 3 = \frac{5x+1}{4}$

18. $\frac{2t-1}{5t} - \frac{2}{3} = \frac{2t+9}{15t}$

8. $\frac{6x+7}{5} - \frac{2x-1}{10} = 3$

19. $\frac{7-2t}{3t} + \frac{5}{2} = \frac{4+t}{6t}$

9. $\frac{2x}{3} - \frac{3}{7} = \frac{4x+5}{7}$

20. $\frac{3t-4}{7t} - \frac{2t+1}{2t} = \frac{2}{14}$

10. $\frac{3}{5} + \frac{2x-7}{4} = \frac{7+4x}{20}$

21. $\frac{5}{6t} - \frac{3}{4} + \frac{1}{3t} = \frac{t+4}{12t}$

11. $\frac{2}{y} + \frac{3}{y} = \frac{5-y}{2y}$

22. $\frac{t+3}{7t} - \frac{5}{6} = \frac{2-t}{2t}$

23. Solve $\frac{8x+37}{18} - \frac{7x-29}{5x-12} = \frac{4x+12}{9}$.

SOLUTION. First combine the fractions with numerical denominators.

$$\frac{8x+37}{18} - \frac{4x+12}{9} = \frac{7x-29}{5x-12}$$

$$\frac{8x+37-8x-24}{18} = \frac{7x-29}{5x-12}$$

$$\frac{13}{18} = \frac{7x-29}{5x-12}$$

Solve:

$$x = 6$$

Check by substituting 6 for x in the original equation.

Solve:

$$24. \frac{3x-2}{2x-5} + \frac{3(x-7)}{5} = \frac{3x-11}{5}$$

$$25. \frac{8x+5}{14} - \frac{3-7x}{6x+2} = \frac{4x+6}{7}$$

$$26. \frac{6x+7}{15} - \frac{2(x-1)}{7x-6} = \frac{2x+1}{5}$$

$$27. \frac{16x+3}{10} - \frac{2x-5}{5x-1} = \frac{8x-1}{5}$$

$$28. \text{ Solve } \frac{3}{x+1} - \frac{2x}{x-1} = -2.$$

SOLUTION.

Clear of fractions: $3(x-1) - 2x(x+1) = -2(x+1)(x-1)$

Perform the indicated operations:

$$3x - 3 - 2x^2 - 2x = -2x^2 + 2$$

Simplify and solve: $x = 5$ Check by substituting 5 for x in the original equation.

Solve:

$$29. \frac{x}{x-2} = 1 - \frac{6}{x+2}$$

$$37. \frac{2x-1}{2x+1} = \frac{3x-2}{3x}$$

$$30. \frac{x-3}{x+9} - \frac{x-5}{x+5} = 0$$

$$38. \frac{x+6}{x+1} = \frac{x+4}{x}$$

$$31. \frac{6x-7}{9x+6} - \frac{5x+5}{12x+8} = \frac{1}{12}$$

$$39. \frac{2-9x}{19-12x} = \frac{6x+7}{8x-7}$$

$$32. \frac{3}{x+3} + \frac{1}{x-3} = \frac{10}{x^2-9}$$

$$40. \frac{x-4}{x-2} = 1 - \frac{1}{x-3}$$

$$33. \frac{x}{8x-16} = \frac{x}{9x-9} + \frac{1}{72}$$

$$41. \frac{x+3}{x-2} = \frac{x-1}{x+2} + \frac{2}{x^2-4}$$

$$34. \frac{x+1}{x-1} = \frac{x-4}{x-5}$$

$$42. \frac{2x-1}{x+2} - \frac{x-3}{x-2} = 1$$

$$35. \frac{x-3}{x+2} = \frac{3x-17}{3x-12}$$

$$43. \frac{3x+1}{x+2} - \frac{x-1}{x+1} = 2$$

$$36. \frac{x-3}{x+2} = \frac{x-4}{x}$$

$$44. \frac{2x-1}{x} + \frac{x}{x+2} = 3$$

Equations with decimal fractions.

EXERCISES

1. Find the value of x correct to the *nearest tenth* from the equation $7.2x - 5.3 = 4.8x + 7.95$.

SOLUTION.

$$7.2x - 5.3 = 4.8x + 7.95$$

Combine like terms:

$$2.4x = 13.25$$

Divide both sides by 2.4:

$$x = 5.52^+$$

The quotient of $13.25 \div 2.4$ is not exact, as there is a remainder after each division. The value of x is required to the nearest tenth, hence the result is 5.5. If the quotient were 5.55 or 5.56, the answer would be 5.6, as 5.56 is nearer to 5.6 than to 5.5.

NOTE. When a result is required correct to a nearest decimal place, find the result correct to one more decimal place than the question requires. If the last figure is 5 or greater, add 1 to the preceding figure. If the last figure is less than 5, drop it. Always give the result correct to just the number of decimal places required.

Find the value of x correct to the nearest tenth:

2. $2.3x = 7.5$

5. $2.12x + 5.84 = 5.32x$

3. $4.13x = 8.97$

6. $3.72x = 1.85x + 3.07$

4. $2.19x = 7.53 - .12x$

7. $7.56x = 34.97 - .38x$

Literal equations are solved by the same methods as are employed for numerical equations.

EXAMPLE. Solve for x the equation $\frac{ax}{c} = ab + \frac{x}{b} - c$.

SOLUTION.

$$\frac{ax}{c} = ab + \frac{x}{b} - c$$

Subtract $\frac{x}{b}$ from both sides:

$$\frac{ax}{c} - \frac{x}{b} = ab - c$$

Combine fractions:

$$\frac{abx - cx}{bc} = ab - c$$

Multiply by bc :

$$(ab - c)x = (ab - c)bc$$

Solve:

$$x = bc$$

CHECK.

$$\frac{abc}{c} = ab + \frac{bc}{b} - c$$

$$ab = ab + c - c$$

$$ab = ab$$

EXERCISES

Solve the following equations :

$$1. \frac{x}{b} + \frac{b}{a} = \frac{x}{a} + \frac{a}{b}$$

$$2. \frac{x}{a} + \frac{x}{b} = \frac{8}{ab}$$

$$3. \frac{x}{n} - \frac{x}{m} = \frac{m+n}{mn}$$

$$4. \frac{x}{a-b} = \frac{1}{a+b}$$

$$5. \frac{x-2a}{bx-4b^2} = \frac{2}{a}$$

$$6. \frac{x+b}{x-b} = \frac{a+b}{a-b}$$

$$7. \frac{a}{x} + \frac{3a}{2x} = \frac{5}{2}$$

$$8. \frac{m}{x} - \frac{n}{x} = 1$$

$$9. \frac{4m}{x-2m} = 2$$

$$10. \frac{a}{x-a} = \frac{b}{x-b}$$

$$11. \frac{x}{m} + \frac{nx}{m^2} = 1 - n^2$$

$$12. \frac{x}{c} - \frac{x}{d} = d - c$$

$$13. \frac{x}{a+b} - a = b$$

$$14. \frac{a}{bx} + \frac{b}{ax} = a^2 + b^2$$

$$15. \frac{x-a^2}{x-b^2} = \frac{a}{b}$$

$$16. \frac{m}{x} = \frac{m+1}{x+1}$$

$$17. \frac{x-1}{x+1} = \frac{1-a}{1+a}$$

$$18. \frac{a}{a+b} - \frac{x}{b-a} = \frac{x}{a}$$

Solve the following formulas for the letters indicated in terms of the remaining letters :

$$19. \frac{1}{f} = \frac{1}{p} + \frac{1}{q} \quad \text{Find } f \text{ and } p.$$

$$20. S = \frac{n}{2}(a+l) \quad \text{Find } a \text{ and } l.$$

$$21. S = \frac{rl-a}{r-1} \quad \text{Find } r \text{ and } a.$$

$$22. f = \frac{mg-t}{m} \quad \text{Find } m \text{ and } t.$$

$$23. R = \frac{gs}{g+s} \quad \text{Find } g \text{ and } s.$$

$$24. T = 2\pi r(h+r) \quad \text{Find } h.$$

Systems of Equations with Fractions

Solve:

$$1. \begin{cases} \frac{x+y}{x-y} = 7 \\ \frac{x+y}{x-y+5} = 2 \end{cases} \quad \text{HINT.} \quad \begin{aligned} x+y &= 7(x-y), & \text{or } 6x-8y &= 0 \\ x+y &= 2(x-y+5), & \text{or } x-3y &= -10 \end{aligned}$$

$$2. \begin{cases} \frac{x}{2} + \frac{y}{3} = 4 \\ 5x - 3y = 2 \end{cases}$$

$$7. \begin{cases} 2r - 3s = 6 \\ \frac{3+2r}{5} - \frac{1+5s}{11} = 2 \end{cases}$$

$$3. \begin{cases} \frac{x-3}{5} = -2y \\ x = 6 - 7y \end{cases}$$

$$8. \begin{cases} 5(2x+y) = 7(x+2y) \\ \frac{6-y}{5} = \frac{3x-2}{7} \end{cases}$$

$$4. \begin{cases} \frac{2x+y}{x-y} = 3 \\ \frac{3x-2y}{y+1} = 5 \end{cases}$$

$$9. \begin{cases} \frac{x}{y} = \frac{4}{5} \\ \frac{x-4}{y+8} = \frac{-4}{21} \end{cases}$$

$$5. \begin{cases} \frac{x+y}{3} - \frac{x-y}{4} = 11 \\ \frac{x+y}{2} - \frac{x-y}{3} = 8 \end{cases}$$

$$10. \begin{cases} \frac{n+3}{d-5} = \frac{8}{7} \\ \frac{n-2}{d+3} = \frac{1}{5} \end{cases}$$

$$6. \begin{cases} \frac{2n+d}{n-d} = \frac{1}{2} \\ \frac{4-n-d}{2n} = \frac{2}{5} \end{cases}$$

$$11. \begin{cases} \frac{x+y-2}{x-y} = \frac{1}{5} \\ \frac{x+2y-3}{x-3y} = -\frac{1}{8} \end{cases}$$

Systems of equations of the form at the right are not linear systems, since they contain terms in xy when they are cleared of fractions; but they may be solved without clearing of fractions by methods similar to those used in solving systems of linear equations.

$$\begin{cases} \frac{a}{x} + \frac{b}{y} = c \\ \frac{d}{x} + \frac{e}{y} = f \end{cases}$$

EXAMPLE. Solve the system:

$$\begin{cases} \frac{9}{x} - \frac{10}{y} = 1 & (1) \end{cases}$$

$$\begin{cases} \frac{12}{x} + \frac{15}{y} = 7 & (2) \end{cases}$$

SOLUTION. Multiply (1) by 3:

$$\frac{27}{x} - \frac{30}{y} = 3 \quad (3)$$

Multiply (2) by 2:

$$\frac{24}{x} + \frac{30}{y} = 14 \quad (4)$$

Add (3) and (4):

$$\frac{51}{x} = 17 \quad (5)$$

Multiply (5) by x :

$$51 = 17x \quad (6)$$

Solve (6):

$$x = 3 \quad (7)$$

Substitute 3 for x in (2):

$$4 + \frac{15}{y} = 7 \quad (8)$$

Solve (8):

$$y = 5 \quad (9)$$

Hence the solution is $\begin{cases} x = 3 \\ y = 5 \end{cases}$

Check by substituting $x = 3$ and $y = 5$ in (1) and (2).

EXERCISES

Solve the following equations. Check the results:

$$1. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{3}{2} \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{2} \end{cases}$$

$$2. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \\ \frac{2}{x} - \frac{1}{y} = \frac{2}{3} \end{cases}$$

$$3. \begin{cases} \frac{1}{x} + \frac{2}{y} = 10 \\ \frac{4}{x} + \frac{3}{y} = 20 \end{cases}$$

$$4. \begin{cases} \frac{9}{x} + \frac{4}{y} = 5 \\ \frac{3}{x} - \frac{8}{y} = -3 \end{cases}$$

$$5. \begin{cases} \frac{2}{x} - \frac{1}{y} = -\frac{1}{6} \\ \frac{9}{x} + \frac{10}{y} = -8 \end{cases}$$

$$6. \begin{cases} \frac{3}{x} + \frac{2}{y} = \frac{13}{16} \\ \frac{1}{x} + \frac{4}{y} = \frac{11}{16} \end{cases}$$

$$7. \begin{cases} \frac{1}{x} + \frac{6}{y} = \frac{9}{8} \\ \frac{4}{x} + \frac{3}{y} = 1 \end{cases}$$

$$8. \begin{cases} \frac{8}{x} + \frac{15}{y} = -1 \\ \frac{10}{x} - \frac{9}{y} = 8 \end{cases}$$

Solve the equations:

$$9. \begin{cases} \frac{9}{x} - \frac{10}{y} = 1 \\ \frac{12}{x} + \frac{15}{y} = 7 \end{cases}$$

$$10. \begin{cases} \frac{1}{x} + \frac{1}{6} = \frac{2}{y} \\ \frac{2}{x} + \frac{3}{y} = 1\frac{5}{12} \end{cases}$$

$$11. \begin{cases} \frac{3}{x} + \frac{1}{y} = 11 \\ \frac{4}{x} + \frac{5}{y} = 22 \end{cases}$$

$$12. \begin{cases} \frac{5}{x} + \frac{10}{y} = \frac{9}{2} \\ \frac{2}{x} - \frac{1}{y} = \frac{4}{5} \end{cases}$$

$$13. \begin{cases} \frac{5}{x} - \frac{2}{y} = 7 \\ \frac{1}{x} + \frac{1}{y} = 0 \end{cases}$$

$$14. \begin{cases} \frac{5}{x} + \frac{6}{y} = 27 \\ \frac{6}{x} + \frac{5}{y} = 28 \end{cases}$$

$$15. \begin{cases} \frac{3}{x} + \frac{8}{y} = 3 \\ \frac{15}{x} - \frac{4}{y} = 4 \end{cases}$$

$$16. \begin{cases} \frac{5}{x} - \frac{2}{y} = -2 \\ \frac{25}{x} + \frac{1}{y} = 6 \end{cases}$$

Literal Systems of Equations

Solve:

$$1. \begin{cases} ax - by = a - b \\ x - y = 0 \end{cases}$$

$$2. \begin{cases} 2ax + 3by = c \\ bx + 2y = b \end{cases}$$

$$3. \begin{cases} ax + by = c \\ bx - ay = d \end{cases}$$

$$4. \begin{cases} ax - by = ab \\ bx + ay = a^2 \end{cases}$$

$$5. \begin{cases} rx + y = r^2 + 1 \\ x + ry = r^2 + 2r - 1 \end{cases}$$

$$6. \begin{cases} \frac{x}{b} + \frac{y}{a} = \frac{a+b}{ab} \\ x = y \end{cases}$$

$$7. \begin{cases} ax + y = b \\ x + ay = a \end{cases}$$

$$8. \begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

$$9. \begin{cases} x + y = 2a \\ (a - b)x = (a + b)y \end{cases}$$

$$10. \begin{cases} ax - by = cd \\ bx + ay = ef \end{cases}$$

$$11. \begin{cases} abx + cdy = cd \\ acx - bdy = bd \end{cases}$$

$$12. \begin{cases} \frac{x}{ac} - \frac{y}{ab} = \frac{1}{bc} \\ cx = by \end{cases}$$

Problems with Fractions

1. Mary had $\frac{2}{3}$ as much money as Jane. Mary spent $\frac{1}{3}$ of her money and Jane spent $\frac{3}{4}$ of her money. Together they had \$.25 left. How much did each have at first?

HINT. Jane had c cents and Mary had $\frac{2}{3}c$ cents. Mary spent $\frac{2}{9}c$ cents and had left $\frac{4}{9}c$ cents. Jane spent $\frac{3}{4}c$ cents and had left $\frac{c}{4}$ cents. Then $\frac{4}{9}c + \frac{c}{4} = 25$. Explain.

2. If $\frac{1}{2}$ of a certain number plus $\frac{1}{3}$ of the number is 48 more than $\frac{1}{6}$ of the number, find the number.

3. The difference between $\frac{1}{4}$ of a certain number and $\frac{1}{3}$ of the number is the same as $\frac{1}{8}$ of the number diminished by 2. Find the number.

4. What number added to both terms of the fraction $\frac{3}{5}$ will give a fraction equal to $\frac{3}{4}$?

5. What number subtracted from both terms of the fraction $\frac{3}{4}$ will give a fraction equal to $\frac{3}{5}$?

6. What number added to both terms of the fraction $\frac{18}{25}$ will make the resulting fraction equal $\frac{4}{5}$?

7. The denominator of a certain fraction exceeds the numerator by 6. One half of the numerator plus $\frac{1}{3}$ of the denominator is 17. Find the fraction.

8. Find a fraction whose value is $\frac{7}{12}$ and whose denominator is 4 less than twice its numerator.

9. What number added to the numbers 3, 8, 12, and 20 will make the sums proportional?

10. What number subtracted from each of the numbers 25 and 29 will make the results proportional to 4 and 5?

11. Two numbers are in the ratio of 3 to 4. If 24 is subtracted from each number, the results are in the ratio of 3 to 5. What are the original numbers?

12. The sum of two numbers is 57. If the greater number is divided by the less, the quotient is 2 and the remainder is 9. Find the numbers.

13. The difference between two numbers is 8. Twice the sum of their reciprocals is equal to 3 times the difference of their reciprocals. Find the numbers.

14. Mr. Corwin has $\frac{2}{5}$ of his property invested at 4%, $\frac{1}{4}$ at 3%, and the remainder at 2%. How much property has he if his income from it is \$915?

15. Miss Halstead made two investments amounting together to \$4400. On one she gained 13% and on the other she lost 6%. If her net gain was \$268, what was the amount of each investment?

16. A man bought eggs at the rate of 3 for \$.10. He sold $\frac{1}{3}$ of them at the rate of 4 for \$.15 and the remainder at the rate of 3 for \$.13. He cleared \$2.90 on the transaction. How many eggs did he buy?

17. A scout master told his troop that $\frac{2}{3}$ of his age 3 yr. ago was equal to $\frac{1}{2}$ his age 6 yr. hence. How old was the scout master?

18. If 6 yr. ago Mary was $\frac{1}{2}$ as old as her teacher and 3 yr. hence Mary will be $\frac{7}{11}$ as old as her teacher, how old is each now?

19. Two quarts of alcohol are mixed with 5 qt. of water. How many quarts of alcohol must be added to the mixture so that the mixture shall be 75% alcohol?

HINT. $\frac{2+a}{7+a} = \frac{75}{100}$. Explain.

20. How much water must be added to 30 lb. of a 10% salt solution so that the resulting solution shall be $7\frac{1}{4}\%$ salt?

21. A 24-pound salt solution is 12% salt. Another solution is 4% salt. How many pounds of the second solution should be added to the first in order to get a solution 10% salt?

22. If 12 lb. of iron weighs 11 lb. in water and 20 lb. of lead weighs 19 lb. in water, find the amount of lead and of iron in a mass that weighs 72 lb. in air and 68 lb. in water.

23. A goldsmith has two alloys of gold, the first being 80% pure gold and the second, $41\frac{2}{3}\%$ pure gold. How much of each must be taken to produce 85 oz. of an alloy that shall be $66\frac{2}{3}\%$ pure gold?

24. An alloy of silver and gold weighing $33\frac{1}{2}$ lb. loses 2 lb. when weighed in water. How many pounds of each metal does it contain if gold, when weighed in water, loses $\frac{1}{9}$ of its weight and silver loses $\frac{1}{10}$ of its weight?

25. Two automobiles start from Chicago and travel north and south respectively. The first car travels at an average speed that is $\frac{2}{3}$ the rate of the other car. If in 5 hr. they are 305 mi. apart, what are their respective rates of travel?

26. A train runs 234 mi. in a certain time. If it ran 6 mi. an hour faster, it could go 39 mi. farther in the same time. Find the rate of the train.

27. A fast freight train runs 200 mi. in the same time that a passenger train goes 375 mi. Find the rate of each train if the freight train goes 21 mi. an hour less than the passenger train.

28. It is noticed that a steamer that runs 14 mi. an hour on a lake makes a run of 72 mi. down a river in the same time that it runs 54 mi. up the river. What is the rate of the river current?

29. John can row 3 mi. an hour and Henry can row 4 mi. an hour in still water. John is 21 mi. farther upstream than Henry and they row toward each other till they meet $13\frac{1}{2}$ mi. below John's starting point. Find the rate of the current.

30. Find each of two complementary angles if they are in the ratio of 2 to 3.

31. Find an angle such that its complement and its supplement have the ratio of 2 to 5.

Work problems.

1. If John can mow the lawn in 2 hr. and Arthur takes 3 hr., how long will it take both of them working together?

HINT. If John can mow the lawn in 2 hr., he can do $\frac{1}{2}$ of it in 1 hr. Similarly, Arthur can do $\frac{1}{3}$ of it in 1 hr. Together they will do $\frac{1}{2} + \frac{1}{3}$, or $\frac{5}{6}$, of the lawn in 1 hr. If h is the number of hours it will take them to mow the lawn working together, then $\frac{1}{h}$ is the part that they can do in 1 hr.

Hence
$$\frac{5}{6} = \frac{1}{h}$$

Solve the equation for h .

2. Edward can do a piece of work in 3 days and Charles takes 4 days. How long will it take them working together?

HINT. The equation is $\frac{1}{d} = \frac{7}{12}$. Explain.

3. James can build a dog house in 6 hr. and George can build it in 8 hr. How long will it take them together?

4. A can lay a certain walk in 12 hr.; B can do the same work in 9 hr.; but C requires 15 hr. How long should it take all three working together?

5. A can do a certain piece of work in 8 days, and B can do the same work in 6 days. After working alone for 2 days, A gets B to help him finish the work. How long does B work?

HINT. Let d = the number of days B works

Then $2 + d$ = the number of days A works

Since A takes 8 days, he can do $\frac{2+d}{8}$ of the work in $(2 + d)$ days.

Since B can do the work in 6 days, he can do $\frac{d}{6}$ of it in d days.

Then
$$\frac{2+d}{8} + \frac{d}{6} = 1$$

Solve for d : $d = 2\frac{2}{3}$

Hence B works $2\frac{2}{3}$ days.

6. Jack can dig a certain trench in 12 days and Paul can dig the same trench in 15 days. Jack works 3 days and is then joined by Paul. How long do they work together?

7. A can do a certain piece of work in 10 days, and A and B working together can do the same work in 6 days. How long does it take B working alone to do the work?

8. A mechanic can overhaul an automobile in 9 hr. and his helper can do the work in 12 hr. The helper begins the job and works 5 hr. Then the mechanic takes the work over and finishes it alone. How long does it take the mechanic?

9. In a factory one machine can turn out a certain number of articles in 6 hr. and a second machine takes 8 hr. for the same work. After they have worked together for $2\frac{1}{2}$ hr., the slower machine develops trouble and is stopped. How long does it take the larger machine to finish the work?

10. A cistern can be filled by a large pipe in 6 hr. and it can be emptied by a smaller pipe in 12 hr. If the cistern is empty and both pipes are then opened and run to full capacity, how long will it take to fill the cistern?

HINT. $\frac{1}{6} - \frac{1}{12} = \frac{1}{h}$. Explain.

11. Alfred and Bruce can build a fence around a field in 8 days. Alfred works alone for 3 days. Then he is joined by Bruce and the two boys finish the work in 6 days. How long would it take each working alone to build the fence?

HINT. The part that Alfred can do in 1 day is $\frac{1}{a}$. The part that Bruce can do in 1 day is $\frac{1}{b}$. Accordingly $\frac{1}{a} + \frac{1}{b} = \frac{1}{8}$. Explain.

Then $\frac{9}{a}$ = the part Alfred does in 9 days and $\frac{6}{b}$ = the part Bruce does in 6 days. Hence $\frac{9}{a} + \frac{6}{b} = 1$, the completed work.

12. James and Ray can build a summer camp in 4 days. James works alone for 3 days and then is joined by Ray. The two boys complete the work in 3 days more. How long would it take each alone to do the work?

13. Mary and Ruth together can make a dress in $4\frac{1}{2}$ hr.; but if Mary works 6 hr. alone, Ruth can complete the dress in 4 hr. How long would it take each alone to make the dress?

14. If Andrew works alone for 4 days on a certain piece of work, Frank can complete the work in 6 days; but if Frank works alone for 3 days, Andrew can complete the work in 6 days. How long will it take each alone to do the work?

15. A and B together can build a boat in 10 days; but at the end of 7 days, A stops working and B finishes the work in 5 days. How long would it take each to do the entire work alone?

16. A tank holding 1000 gal. can be emptied in 20 min. by 2 pipes, A and B, running together. Both pipes run together 8 min., then A is stopped, and the tank is emptied by B in 25 min. more. How many gallons a minute does each pipe carry?

17. Two pipes empty water into a cistern. If pipe A runs 6 hr. and pipe B runs 9 hr., the cistern is $\frac{3}{4}$ full; but if pipe B runs 6 hr., then pipe A can fill the cistern in 16 hr. How long would it take each pipe alone to fill the cistern?

Problems of investment.

Before working these problems you should recall the meaning of such terms as *stocks*, *bonds*, *dividends*, *interest*, etc.

In the problems that follow, all stocks and bonds are bought at par. The par value of bonds is considered as \$1000, and that of stocks, as \$100.

Problems of investment and income are most important and require careful study.

1. What does the word *amount* mean when used in connection with interest? What is the formula for the amount of p dollars at r per cent simple interest for t years?

2. A man invested \$3000. He put d dollars at 5% and the rest at 6%. What was his annual income?

3. If stock costs \$25 a share and pays an annual dividend of \$2, what is the per cent of yield? What is the yield if it pays \$1.50 a year?

4. Mrs. Field invested \$3600. With part of this amount she purchased bonds that paid 6% and the rest she deposited in a savings bank that paid $4\frac{1}{2}\%$. If her annual income from this investment was \$207, how much money was placed at each rate?

SOLUTION.

Let p = the number of dollars at $4\frac{1}{2}\%$
and $3600 - p$ = the number of dollars at 6%

Then $\frac{4\frac{1}{2}}{100}p$ = the income from bank at $4\frac{1}{2}\%$

and $\frac{6}{100}(3600 - p)$ = the income from bonds at 6%

$$\frac{4\frac{1}{2}}{100}p + \frac{6}{100}(3600 - p) = 207, \text{ the total income}$$

Multiply both sides by 100:

$$4\frac{1}{2}p + 6(3600 - p) = 20,700$$

Solve for p :

$$p = 600$$

$$3600 - p = 3000$$

Hence Mrs. Field placed \$600 at $4\frac{1}{2}\%$ and \$3000 at 6%.

5. Mr. Lyons received \$270 yearly from his investment of \$5000. If part of this was placed at 5% and the rest at 6%, find the amount at each rate.

6. Mr. Righter invested \$30,000 in bonds and received an income that averaged \$155 a month. If he received 6% on part and $6\frac{1}{2}\%$ on the rest, how much did he have invested at each rate?

7. Mrs. Harper invested some money in stocks that paid 7% dividend. She put 3 times this amount in bonds that paid 5%. If her annual income was \$440, how much did she have invested in the bonds?

8. Mr. Bond invested $\frac{1}{3}$ of his money at 4%, $\frac{1}{2}$ of it at 5%, and the remainder at 7%. If his average income was \$100 a month, how much money did he invest?

9. Miss Potter received a legacy of \$25,000. She invested part at 6%, an equal amount at 5%, and the remainder at 7%. If her total income was \$1480, how much did she have invested at each rate?

10. Mr. Frank has \$3200, part in a savings bank at $4\frac{1}{3}\%$ interest and the rest in a mortgage at 6%. If his income from the mortgage is 4 times the interest from the bank, what is the size of the mortgage?

11. Mrs. Adams has \$10,000 invested at 5% and \$8000 at 6%. How much must she put at 7% so that her entire income may be 6% of her total investment?

HINT. $500 + 480 + \frac{7}{100}n = \frac{6}{100}(18,000 + n)$. Explain.

12. A man invests \$12,000 so as to receive an average income of $5\frac{1}{2}\%$. Part of his money is invested at $4\frac{1}{2}\%$ and the rest at 6%. How much has he invested at 6%? at $4\frac{1}{2}\%$?

13. Miss Snyder received an income of \$204 from her investments. If $\frac{1}{3}$ of her investments yielded 5% and the rest, 6%, how much did she have invested?

14. Mr. Goodrich received the same income from an investment of \$6000 that he did from an investment of \$3600 at a rate 3% higher than the first rate. What rates of interest did he receive?

15. Mr. Burt bought bonds that paid $5\frac{1}{2}\%$. Mr. Woodard invested \$1000 less in stock that yielded 6%. If Mr. Burt's income was \$35 greater, how much did each receive?

16. Mr. Anderson had a part of his money in a bank that paid 5% interest and \$3500 more than that amount in a mortgage that paid 6%. If his total income from these investments was \$375, how much money did he have?

17. Miss Sharp received the same income from a $5\frac{1}{2}\%$ investment that she did from a 7% investment. The amount at 7% was \$900 less than that at $5\frac{1}{2}\%$. How much did she have in these two investments?

Mixture problems.

1. A merchant sold 55 lb. of mixed nuts at \$.35 a pound at a profit of \$.35 more than 40% of the cost. Some of the nuts cost \$.20 a pound and the rest, \$.30 a pound. How many pounds of each did he have in the mixture?

HINT. Let n = the number of pounds of 30-cent nuts
and $55 - n$ = the number of pounds of 20-cent nuts

Then $30n + 20(55 - n) = 10n + 1100$, the cost of the nuts

$\frac{40}{100}(10n + 1100) + 35 = 4n + 475$, the profit

$$(10n + 1100) + (4n + 475) = 1925$$

Explain the equation and complete the solution.

2. A confectioner mixed 3 grades of candy to sell at \$.75 a pound. He had some 70-cent candy, $\frac{1}{3}$ more 50-cent candy than 70-cent candy, and 3 lb. more of dollar candy than of 50-cent candy. How many pounds of each grade did he have?

3. One cask contains a mixture of 14 gal. of alcohol and 16 gal. of water. Another contains 20 gal. of alcohol and 10 gal. of water. How many gallons must be drawn from each to make a 10-gallon mixture having half alcohol?

4. If 1 cu. in. of aluminum weighs .092 lb. and 1 cu. in. of copper weighs .31 lb., what is the percentage of composition by weight of an alloy of the metals that weighs .276 lb. to the cubic inch?

5. A florist had 50 lb. of grass seed for sale at \$.30 a pound. Some of the seed cost him \$.15 a pound and the rest, \$.25 a pound. He added, for expenses and profit, $33\frac{1}{3}\%$ of the original cost. How much of each grade of seed did he have?

6. Jane and Harriet conducted the candy booth at a church fair. Their sales amounted to \$25.60 and they cleared \$.40 more than 50% of the cost of the candy. If they had some 60-cent candy mixed with 3 times as much 50-cent candy, how much of each grade did they have?



Fun with Figures



1. Think of any number, double it, add 12, divide by 2, and subtract the original number. Your answer is 6. Explain.
2. Think of a number, add to it $\frac{1}{3}$ of itself, divide the result by 4, then multiply by 3. Why is the result the same as the original number?
3. Think of another number, multiply it by 6, increase the result by 9, then divide the sum by 3, and subtract 3. Why is the result twice the original number?
4. Can a fraction whose numerator is less than its denominator be equal to a fraction whose numerator is greater than its denominator? Explain.
5. If a chicken weighs 24 oz. and a half a chicken, how much does a chicken and a half weigh?
6. Show that the product of any two consecutive even integers can be exactly divided by 8.
7. Kenneth said to Arthur, "Give me \$10; then I shall have as much as you." Arthur replied, "You give me \$10; then I shall have twice as much as you." How much did each have?
8. What is the matter with this "proof"?

Let $a = b = 5$. Then $a^2 = b^2$ and $a^4 = b^4$

$$a^2 - b^2 = 0 \quad \text{and} \quad a^4 - b^4 = 0$$

Then $a^2 - b^2 = a^4 - b^4$, since each equals 0

$$\therefore a^2 - b^2 = (a^2 - b^2)(a^2 + b^2) \quad \text{Why?}$$

Divide by $a^2 - b^2$: $1 = a^2 + b^2 = 25 + 25$

or $1 = 50$

9. Do you agree with the following "proof"?

(a) Let f = your father's age, y = your age, and $f - y = d$.

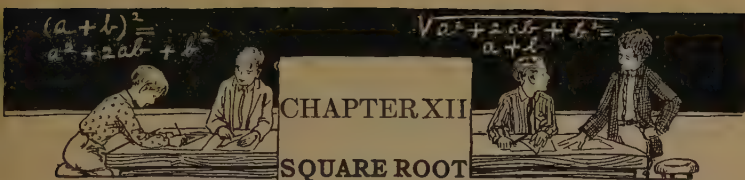
$$\text{Then } f = y + d$$

Multiply by $f - y$: $f^2 - fy = fy + fd - y^2 - yd$

$$f^2 - fy - fd = fy - y^2 - yd$$

Divide by $f - y - d$: $f = y$

Or your age is the same as your father's.



What is meant by the square root of an expression? (See page 156.) Since $+3 \times +3 = 9$ and $-3 \times -3 = 9$, it follows that the square root of 9 is $+3$ or -3 ; that is, the square root of an expression may be $+$ or $-$. The positive square root is called the **principal square root**. This is the root indicated when no sign precedes the radical sign; as, $\sqrt{4} = 2$. The negative root is indicated by $-\sqrt{4}$, which equals -2 .

The square root of a monomial composed of literal factors is found by dividing the exponents of the literal factors by 2. Similarly, any root of a literal term is found by dividing the exponents of the literal factors by the index of the root.

Thus, $\sqrt{a^2b^6} = a^{\frac{2}{2}}b^{\frac{6}{2}} = ab^3$; $\sqrt[3]{a^6b^9} = a^{\frac{6}{3}}b^{\frac{9}{3}} = a^2b^3$; and $\sqrt[4]{a^4b^{16}} = a^{\frac{4}{4}}b^{\frac{16}{4}} = ab^4$.

Any root of a monomial is obtained by using the root of the numerical factor as the coefficient of the same root of the literal factors.

Thus, $\sqrt{16 a^2b^8} = 4 ab^4$ and $\sqrt[3]{27 x^3y^6} = 3 xy^2$.

The root of a product is equal to the product of the corresponding roots of the factors.

Thus, $\sqrt{36} = \sqrt{4} \times \sqrt{9}$ and $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$.

ORAL EXERCISES

Find the indicated roots:

1. $\sqrt{144 x^2y^8}$

4. $\sqrt{49 a^8b^2c^4}$

7. $\sqrt[3]{8 a^6b^9}$

2. $\sqrt{64 a^4b^8}$

5. $\sqrt{100 a^2b^2c^6}$

8. $\sqrt[3]{27 a^3b^6c^9}$

3. $\sqrt{81 m^6n^2}$

6. $\sqrt{121 r^2s^8}$

9. $\sqrt[3]{125 x^6y^9}$

Square Root of Arithmetic Numbers by Factoring

The square root of larger numbers sometimes may be found by separating the numbers into their prime factors and finding the product of the square roots of the factors.

$$\text{Thus, } \sqrt{225} = \sqrt{3 \times 3 \times 5 \times 5} = \sqrt{3^2 \times 5^2} = 3 \times 5 = 15.$$

$$\text{Similarly, } \sqrt{6561 a^6} = \sqrt{9 \times 9 \times 9 \times 9 a^6} = \sqrt{3^8 a^6} = 3^4 a^3 = 81 a^3$$

EXERCISES

Find the square roots of the following and check:

- | | | | |
|--------|---------|---------------|-----------------------------|
| 1. 576 | 3. 5184 | 5. 1764 | 7. $10 \times 18 \times 20$ |
| 2. 324 | 4. 1296 | 6. $2304 a^4$ | 8. $85 \times 15 \times 51$ |

9. Find the square root of $\frac{18 \times 10 \times 45}{15 \times 30 \times 98}$.

$$\text{SOLUTION. } \sqrt{\frac{18 \times 10 \times 45}{15 \times 30 \times 98}} = \sqrt{\frac{9}{49}} = \frac{3}{7}$$

Find the square roots of:

- | | |
|---|---|
| 10. $\frac{15 \times 10 \times 24}{98 \times 88 \times 11}$ | 11. $\frac{15 \times 21 \times 35}{25 \times 75 \times 12}$ |
|---|---|

$$12. \frac{30 \times 10 a^2 \times 60 b^2}{35 \times 21 a \times 12 a x^2}$$

Square Root of Arithmetic Numbers

Since the square root of many numbers cannot be found by factoring, a more general method must be learned.

Thus, since $25 = 5 \times 5$, it is easily seen that $\sqrt{25} = 5$; but $\sqrt{23}$ or $\sqrt{25.2}$ cannot be found so readily.

Note the number of digits in the following squares:

$$\begin{array}{l} .01^2 = .00'01 \quad | \quad .1^2 = .01 \quad | \quad 1^2 = 1 \quad | \quad 10^2 = 1'00 \quad | \quad 100^2 = 1'00'00 \\ .09^2 = .00'81 \quad | \quad .9^2 = .81 \quad | \quad 9^2 = 81 \quad | \quad 99^2 = 98'01 \quad | \quad 999^2 = 99'80'01 \end{array}$$

A one-digit or a two-digit square has one digit in its square root. Thus, $\sqrt{1} = 1$, $\sqrt{81} = 9$, $\sqrt{.01} = .1$.

A three-digit or a four-digit square has two digits in its square root. Thus, $\sqrt{100} = 10$, $\sqrt{9801} = 99$, $\sqrt{.0081} = .09$.

A five-digit or a six-digit square has three digits in its square root. Thus, $\sqrt{10000} = 100$, $\sqrt{998001} = 999$.

If a number is separated into groups of two digits each, beginning at the decimal point, the square root of the number will have as many digits as there are groups.

The formula $(t + u)^2 = t^2 + 2tu + u^2 = t^2 + u(2t + u)$ provides a way to find the square root of numbers.

Let any number of two digits, as 73, be represented by $t + u$, in which $t = 70$ and $u = 3$. Then

$$(t + u)^2 = t^2 + 2tu + u^2, \text{ or } t^2 + u(2t + u)$$

$$73^2, \text{ or } (70 + 3)^2 = 70^2 + 2(70)(3) + 3^2, \text{ or } 70^2 + 3[2(70) + 3]$$

EXAMPLE 1. Find the square root of 5329.

SOLUTION.		53 29	<u>70 + 3</u>	CHECK. 73
	$t^2 = 70^2 =$	49 00		73
	$2t = 2 \times 70 = 140$	4 29		219
	$u =$	3		511
	$2t + u =$	143		5329
	$u(2t + u)$	= 429		

The square root of 5329 is ± 73 .

First separate 5329 into groups of two digits each, beginning at the decimal point, to determine the number of digits in the result. This gives 53'29, or $5300 + 29$.

The largest square included in 5300 is 4900 and its square root is 70. Accordingly, 70 is the first part of the result. Subtract 70^2 , or 4900, from 5329. The remainder is 429.

Find the next trial divisor by taking twice the part of the root already found: $2 \times 70 = 140$. Divide the preceding remainder, 429, by 140 to find the probable next figure of the root. This division gives 3 as the probable last figure. The sum of 140 and 3 corresponds to $2t + u$ of the formula.

Subtract the product of 3×143 from 429. As the remainder is 0, the work is complete. The work may be checked by squaring 73.

In practice, the necessary work may be abbreviated thus:

Several zeros are omitted, as they serve simply to locate the position of the other digits. This position is given in the work.

$$\begin{array}{r}
 5329 \quad \boxed{73} \\
 t^2 = \quad 49 \\
 2t + u = 143 \quad \boxed{429} \\
 u(2t + u) = \quad \boxed{429}
 \end{array}$$

EXAMPLE 2. Find the square root of 800.89.

SOLUTION.

$$\begin{array}{rcl}
 & & 2 \ 8.3 \\
 & & \overline{) 800.89} \\
 (1) & t^2 = & 4 \\
 (2) & \begin{cases} 2t + u = 48 \\ u(2t + u) = 384 \end{cases} & \begin{array}{r} 400 \\ 384 \\ \hline 1689 \end{array} \\
 (3) & \begin{cases} 2t + u = 563 \\ u(2t + u) = 1689 \end{cases} & \begin{array}{r} 1689 \\ 1689 \\ \hline \end{array}
 \end{array}$$

The explanation is similar to that for ex. 1.

Notice the divisor in step (2), $2t = 40$. Apparently 40 is contained in the remainder, 400, 9 times or 10 times; but if these numbers are tried, it is found that they are too large and that 8 is the correct quotient.

Step (3) is a continuation of the method of step (2). In step (3), $t = 280$.

NOTE 1. The left-hand group of digits may contain only one digit, but the right-hand group must always contain two digits. If the right-hand group is at the right of the decimal point and contains only one digit, a zero must be annexed to make a complete group of two digits. Thus, 389.756 becomes 3'89'.75'60.

NOTE 2. The square root of a fraction may be found by finding the square root of the numerator and the square root of the denominator of the fraction and forming a new fraction with the results; or the fraction may be reduced to a decimal and the square root of the decimal may be found.

NOTE 3. The square roots of numbers that are not exact squares are found to the nearest tenth or to the nearest hundredth, etc. If the square root to the nearest hundredth is required, the square root is found correct to thousandths (see page 206). If the last figure is less than 5, it should be dropped and the preceding figure should be retained as the final one. If, however, the thousandth figure is 5 or greater, it should be dropped and the preceding figure should be increased by 1. Thus, 2.864 becomes 2.86 and 2.865 or 2.868 becomes 2.87.

EXERCISES

Find the square roots of the following:

- | | | |
|---------|------------|------------|
| 1. 1156 | 4. 3721 | 7. 264,196 |
| 2. 2401 | 5. 2809 | 8. 82,369 |
| 3. 4761 | 6. 168,100 | 9. 150,544 |

10. State the steps to be followed in finding the square root of a number.

Find the square roots of the following :

11. 79.21

14. 97.4169

17. $\frac{225}{361}$

12. 43.56

15. $\frac{576}{878}$

18. $\frac{841}{1521}$

13. 1.5129

16. $\frac{961}{1444}$

19. $\frac{289}{729}$

Find, to the nearest hundredth, the square root of :

20. 2

24. 9.871

28. $\frac{5}{8}$

21. 3

25. 15.045

29. $\frac{7}{8}$

22. 5

26. 285.3

30. $\frac{2}{3}$

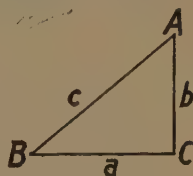
23. 7

27. 17.952

31. $\frac{4}{5}$

You have learned that in a right triangle (a triangle containing an angle of 90°), the side opposite the right angle is called the **hypotenuse**.

It is proved in geometry that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. That is, $c^2 = a^2 + b^2$. This is known as the **law or theorem of Pythagoras**.



32. Find the hypotenuse of a right triangle if $a = 12$, $b = 5$; if $a = 27$, $b = 36$.

33. If the hypotenuse of a right triangle is 72 in. and one of the other sides is 65 in., find the remaining side.

34. The area of a right triangle is equal to one half the product of the two shorter sides. Explain.

35. Find the area of a right triangle if the hypotenuse is 26 in. and one of the other sides is 10 in.

36. The area of a circle is given by the formula $A = \pi r^2$. Find the radius of a circular tray if the area of the tray is 100 sq. in. Give the result correct to the nearest tenth of an inch.

37. What is the diameter of a wheel if the area of one side is 700 sq. in.? ($A = \frac{\pi d^2}{4}$)

Tables of Powers and Roots

It is often convenient to use tables similar to that on page 227 in finding powers and roots of numbers. This table gives the squares and the cubes of all numbers from 1 to 100, and the square roots and the cube roots of the same numbers, correct to 3 decimal places.

ORAL EXERCISES

Explain how the following results are obtained from the table:

1. $29^2 = 841$

6. $29^3 = 24,389$

2. $2.9^2 = 8.41$

7. $2.9^3 = 24.389$

3. $.29 = .0841$

8. $.29^3 = .024389$

4. $\sqrt{29} = 5.385$

9. $\sqrt[3]{29} = 3.072$

5. $\sqrt{.29} = .5385$

10. $\sqrt[3]{.029} = .3072$

11. Read from the table the squares of the following numbers: (a) 48; (b) 4.8; (c) .48; (d) .073; (e) 77.

12. What is the square root of: (a) 1296? (b) 13.69? (c) .6724? (d) 7921? (e) 32.49?

13. What is the cube of: (a) 17? (b) 31? (c) 83? (d) 98?

14. What is the cube root of: (a) 63? (b) 75? (c) 19,683? (d) 493.039?

15. Show that the approximate square of 6.782 is 46; that the approximate cube of 3.684 is 50. Explain.

16. What is the approximate value of: (a) 7.937^2 ? (b) 4.02^3 ? (c) 4.53^3 ? (d) 8.602^2 ? (e) 5.2^2 ? (f) 3.9^3 ?

17. Find to the nearest tenth of a rod the diagonal distance across a rectangular lot if its dimensions are 5.9 rd. by 7.8 rd.

18. Find to the nearest hundredth of an inch the edge of a cube if the volume is 80 cu. in.

TABLE OF POWERS AND ROOTS

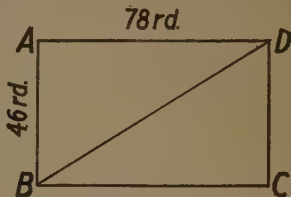
No.	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS	No.	SQUARES	CUBES	SQUARE ROOTS	CUBE ROOTS
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	60	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	70	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.035	78	6,084	474,552	8.831	4.270
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.298
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.326
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.304
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.327
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.364
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.372
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.399
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.416
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.434
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.441
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	90	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.641

Problems from Geometry

1. What is the hypotenuse of a right triangle if the other two sides are 5.4 in. and 7.2 in.?

2. Find the remaining side of a right triangle if the hypotenuse is 6 in. and one of the other sides is 4.8 in. What is the area of this triangle?

3. Find the distance from one corner of a rectangular field to the opposite corner if the sides of the field are 78 rd. and 46 rd.



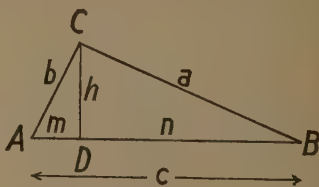
4. What is the area of a rectangular field if the diagonal BD is 52 rd. and one side is 36 rd.?

5. It took 244 rd. of fencing for a rectangular field. Find the diagonal of the field if the sides are in the ratio 19 to 42.

6. The altitude on the hypotenuse of a right triangle divides the hypotenuse into two parts such that $\frac{m}{h} = \frac{h}{n}$. Hence $h^2 = mn$ and

$h = \sqrt{mn}$. Also, $\frac{c}{b} = \frac{b}{m}$. Hence

$b^2 = cm$ and $b = \sqrt{cm}$. Find the altitude if the hypotenuse is 17 in. and $m = 3\frac{1}{4}$ in.



7. Find side b of the triangle in ex. 6. Then find side a .

8. The altitude on the hypotenuse of a certain right triangle divides the hypotenuse into two parts such that $m = 144$ ft. and $n = 25$ ft. Find: (a) the altitude of the triangle; (b) the two sides AC and CB .

Square Root of Polynomials

(Optional Topic)

You will recall that $a^2 + 2ab + b^2 = (a + b)^2$, or $[-(a + b)]^2$. (See pages 156 and 170.) Accordingly, the square root of $a^2 + 2ab + b^2$ is $\pm(a + b)$. A study of this relation will help you to find the square root of other algebraic expressions.

The expression $a^2 + 2ab + b^2$ may be grouped to show the relation $a^2 + b(2a + b)$. That is, a^2 is the square of the first term and the remainder is the product of b by $2a + b$.

The work may be arranged as below.

$$\begin{array}{r} \text{The square of } a = \begin{array}{l} a^2 + 2ab + b^2 \mid \pm(a + b) \\ a^2 \\ \hline b(2a + b) = \end{array} \begin{array}{l} 2ab + b^2 \end{array} \end{array}$$

We know in advance what the result is to be, just as we know that the square root of 9 is ± 3 . Using this knowledge, we are able to arrive at a method of finding the square root of more complicated expressions.

EXAMPLE 1. Find the square root of $9x^2 + 30xy + 25y^2$.

SOLUTION.

$$\begin{array}{r} \text{The square of } 3x = \begin{array}{l} 9x^2 + 30xy + 25y^2 \mid \pm(3x + 5y) \\ 9x^2 \\ \hline 30xy + 25y^2 \end{array} \\ \text{Trial divisor} = 2(3x) = 6x \\ \text{Complete divisor} = 6x + 5y \\ 5y(6x + 5y) = \end{array} \begin{array}{l} 30xy + 25y^2 \end{array}$$

Check by multiplication: $[\pm(3x + 5y)]^2 = 9x^2 + 30xy + 25y^2$

The square root of the first term is $3x$, corresponding to a in $(a + b)^2$.

$2(3x) = 6x$, corresponding to $2a$ in $a^2 + b(2a + b)$.

$\frac{30xy}{6x} = 5y$, corresponding to $\frac{2ab}{2a}$, or b , in $a^2 + b(2a + b)$.

$6x + 5y$ corresponds to $2a + b$ in $a^2 + b(2a + b)$.

$5y(6x + 5y)$ corresponds to $b(2a + b)$ in $a^2 + b(2a + b)$.

EXAMPLE 2. Find the square root of $4x^2 + 12xz - 6yz - 4xy + 9z^2 + y^2$.

SOLUTION. Arrange the terms according to the descending powers of x and continue the method developed in example 1.

$$\begin{array}{rcl}
 (1) \quad (2x)^2 & = & \begin{array}{|l} 4x^2 - 4xy + 12xz - 6yz + y^2 + 9z^2 \end{array} \overline{) \pm (2x - y + 3z)} \\
 (2) \quad -y(4x - y) & = & \begin{array}{|l} -4xy \qquad \qquad \qquad + y^2 \end{array} \\
 (3) \quad 3z(4x - 2y + 3z) & = & \begin{array}{|l} 12xz - 6yz \qquad + 9z^2 \\ 12xz - 6yz \qquad + 9z^2 \end{array}
 \end{array}$$

The trial divisor in (2) is found by adding $\frac{-4xy}{4x}$, or $-y$, to $2(2x)$, giving $4x - y$.

The trial divisor in (3) is found by adding $\frac{12xz}{4x}$, or $3z$, to $2(2x - y)$, or $4x - 2y$, giving $4x - 2y + 3z$.

EXERCISES

Find the square roots of the following polynomials. Check ex. 1 to 10:

1. $x^2 + 8xy + 16y^2$

5. $144x^2 - 216xy + 81y^2$

2. $a^2 - 18ab + 81b^2$

6. $9a^4 - 6a^2b^2 + b^4$

3. $225x^6 - 120x^3y + 16y^2$

7. $9b^4 - 30b^2a + 25a^2$

4. $49a^2 + 28ab^3 + 4b^6$

8. $1 + 6x + 5x^2 - 12x^3 + 4x^4$

9. $4x^4 + 17x^2 - 12x - 12x^3 + 4$

10. $4x^4 + 13x^2 - 12x^3 - 6x + 1$

11. $9x^2 - 12xy + 6x + 4y^2 - 4y + 1$

12. $4x^4 - 4x^2y - 12x^2 + y^2 + 6y + 9$

13. $36a^4 + 12a^3 + 9 - 35a^2 - 6a$

14. $4a^2 - \frac{4ab}{3} + \frac{b^2}{9}$ Ans. $\pm \left(2a - \frac{b}{3}\right)$

15. $\frac{a^2}{4} - 2ab + 4b^2$

17. $a^4 - a^3 + \frac{a^2}{4} + 4a - 2 + \frac{4}{a^2}$

16. $\frac{a^2}{4} + \frac{ab}{3} + \frac{b^2}{9}$

18. $4a^4 + 20a^2 - 3 - \frac{70}{a^2} + \frac{49}{a^4}$

Show that
 $\sqrt{12} = 2\sqrt{3}$

Find the
 value of $\frac{2}{\sqrt{3}}$

CHAPTER XIII

RADICALS AND EXPONENTS

The sign $\sqrt{\quad}$, which, as you have learned, indicates the square root of a quantity, is called the **radical sign**, from the Latin *radix*, which means a *root*. Any expression affected by a radical sign is, for the same reason, called a **radical**. A radical is therefore the indicated root of a quantity.

Thus, $\sqrt{9}$, $\sqrt{a}$, $\sqrt[3]{b}$, and $\sqrt[4]{c}$ are radicals.

The small number written in the radical sign, as the 3 in $\sqrt[3]{b}$, is called the **index** of the root. If no index is expressed, 2 is understood.

The number or expression under the radical sign is called the **radicand**.

Thus, in $\sqrt{5}$ and $\sqrt{a}$, 5 and a are radicands.

Any integer or the quotient of two integers is called a **rational number**.

Thus, 2, -2 , $2\frac{1}{3}$, and $\frac{5}{8}$ are rational numbers.

A radical is **rational** if its root can be found exactly.

Thus, $\sqrt{9}$, $\sqrt[3]{8}$, and $\sqrt{a^2 + 2ab + b^2}$ are rational since their roots are 3, 2, and $a + b$.

A radical is **irrational** if its root cannot be found exactly. Irrational radicals are sometimes called **surds**.

Thus, $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2a + b}$ are irrational radicals or surds, since these roots cannot be found exactly.

The **order** of a surd or of a radical is determined by the index of the indicated root. If two surds have the same index, they are said to be of the *same order*. Thus, $\sqrt{5}$ and $\sqrt{ax}$ are of the second order and are called **quadratic surds**; $\sqrt[3]{2}$ and $\sqrt[3]{b}$ are of the third order and are called **cubic surds**.

Graphic Representation of Irrational Numbers

As you have seen, irrational numbers cannot be stated exactly without the aid of radical signs or some other indicated expression; but some such numbers can be exactly expressed graphically by the lengths of lines. (See page 225.)

If the sides of an isosceles right triangle (Fig. 1) are each 1 unit long, then the hypotenuse of the triangle is $\sqrt{2}$ units long. If the hypotenuse is 2 units long and one side is 1 unit long, then the third side is $\sqrt{3}$ units long, as in Fig. 2.

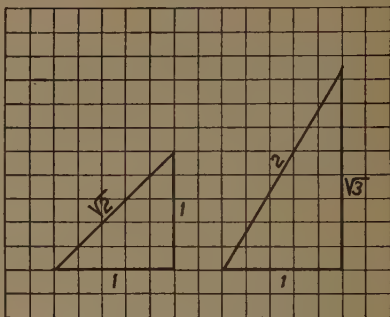


FIG. 1

FIG. 2

EXERCISES

1. What is the length of the hypotenuse of a right triangle if the other two sides are 1 in. and 2 in., respectively?
2. Construct a line equal to $\sqrt{5}$ in. long.
3. If the hypotenuse of a right triangle is 4 in. and one of the other sides is 3 in., what is the remaining side?
4. Construct a line equal to $\sqrt{7}$ in. long.
5. Construct a line equal to $\sqrt{10}$ in. long.

State the index of the root and the radicand in each of the following:

- | | | | |
|---------------|-------------------|---------------------|---------------------|
| 6. $\sqrt{4}$ | 8. $\sqrt[3]{8}$ | 10. $\sqrt[3]{x^4}$ | 12. $\sqrt{a^2}$ |
| 7. $\sqrt{9}$ | 9. $\sqrt[3]{25}$ | 11. $\sqrt[4]{5}$ | 13. $\sqrt[3]{b^3}$ |

14. State which of the radicals in ex. 6 to 13 are irrational.

15. What is the value of the expression in ex. 6? in ex. 7? in ex. 8? in ex. 12? in ex. 13?

Reducing Radicals

If you are to work with expressions that have radical quantities, you must first express them in the simplest form.

A radical is expressed in reduced or simplified form when:

- (a) *The radicand contains no rational factor.*
- (b) *The radicand is integral.*
- (c) *The index of the root is as small as possible.*

Study the following examples to see how radicals having integral radicands are reduced or simplified.

EXAMPLE 1. Express $\sqrt{18}$ in a reduced form.

SOLUTION. $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

Since the square root of a product equals the product of the square roots of the factors (see page 221), $\sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$.

The factor 9 of the product 9×2 is called the **rational factor**.

EXAMPLE 2. Reduce $3\sqrt{8a^3b}$ to simpler form.

SOLUTION. $3\sqrt{8a^3b} = 3\sqrt{4 \times 2a^2ab} = 3\sqrt{4a^2(2ab)} = 6a\sqrt{2ab}$

EXAMPLE 3. Reduce $\sqrt[3]{54a^4b^2}$ to simpler form.

SOLUTION. $\sqrt[3]{54a^4b^2} = \sqrt[3]{27 \times 2a^3ab^2} = \sqrt[3]{27a^3(2ab^2)} = 3a\sqrt[3]{2ab^2}$

Summary of steps.

1. Separate the radicand into the product of two factors, one of which is its largest rational factor.
2. Find the required root of the rational factor.
3. Multiply the coefficient of the radicand, if any, by the root of the rational factor and annex the irrational factor.

EXERCISES

Remove all rational factors from the following radicands:

- | | | | |
|----------------|-------------------|---------------------|------------------------|
| 1. $\sqrt{20}$ | 5. $\sqrt{50}$ | 9. $\sqrt[3]{135}$ | 13. $\sqrt{18ab^2}$ |
| 2. $\sqrt{28}$ | 6. $\sqrt[3]{16}$ | 10. $\sqrt[3]{250}$ | 14. $\sqrt{72\pi r^2}$ |
| 3. $\sqrt{27}$ | 7. $\sqrt[3]{24}$ | 11. $\sqrt{8a}$ | 15. $\sqrt{\pi r^2 h}$ |
| 4. $\sqrt{32}$ | 8. $\sqrt[3]{81}$ | 12. $\sqrt{9a^2b}$ | 16. $3\sqrt[3]{32}$ |

Remove all rational factors :

- | | | |
|-------------------------------|-------------------------------|-------------------------------|
| 17. $5 a \sqrt[3]{72 a^3}$ | 23. $5 \sqrt[3]{375}$ | 29. $5 a \sqrt{242 ab^4}$ |
| 18. $3 ab \sqrt[3]{250 a}$ | 24. $17 a^3 \sqrt{200 b^2 x}$ | 30. $7 a^2 \sqrt[3]{686 a^5}$ |
| 19. $5 xy^2 \sqrt{50 x}$ | 25. $5 m \sqrt{99 n}$ | 31. $\sqrt{(a+b)^2 x}$ |
| 20. $3 a \sqrt{98 a^2}$ | 26. $\sqrt{250 a^4 b}$ | 32. $\sqrt[3]{(a+b)^3 y}$ |
| 21. $5 \sqrt{27 a^3}$ | 27. $2 \sqrt{288 x^3 y^2}$ | 33. $\sqrt{x^3 + x^2}$ |
| 22. $6 \sqrt[3]{128 a^2 b^4}$ | 28. $12 \sqrt{72 x^5}$ | 34. $\sqrt{80(a^4 - a^2)}$ |
| 35. $\sqrt{12 x^3(x-y)^2}$ | 36. $\sqrt[3]{24 a^2(x-y)^3}$ | |

You may find it useful at times to write an expression like $3\sqrt{2}$ so that the entire number shall be under the radical sign, thus:

$$3\sqrt{2} = \sqrt{3^2} \cdot \sqrt{2} = \sqrt{9 \times 2} = \sqrt{18}$$

EXERCISES

Write each of the following expressions so that the entire expression shall be under the radical sign :

- | | | |
|----------------|--------------------|----------------------|
| 1. $2\sqrt{3}$ | 4. $2 a \sqrt{x}$ | 7. $3 a \sqrt[3]{5}$ |
| 2. $2\sqrt{5}$ | 5. $3 b \sqrt{a}$ | 8. $5 \sqrt{7 x}$ |
| 3. $3\sqrt{3}$ | 6. $5 c \sqrt{ab}$ | 9. $7 \sqrt{\pi r}$ |

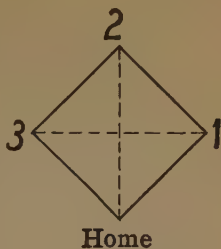
Using Geometric Facts

Express the results to the following in simplest radical form.

1. If the school football field is 300 ft. by 150 ft., what is the longest run in a straight line that a player could make on the field? Express the result: (a) in simplest radical form; (b) to the nearest tenth of a foot.

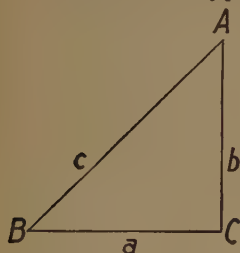
2. What is the length of a diagonal path across a city lot that is 200 ft. by 800 ft.? Express the result: (a) in simplest radical form; (b) to the nearest tenth of a foot.

3. A baseball diamond is a square 90 ft. on a side. What is the distance from the home plate to second base? from first base to third base?



4. The length of a rectangular field is 35 rd. and the diagonal is 45 rd. Find the width of the field.

5. Find the hypotenuse of an isosceles



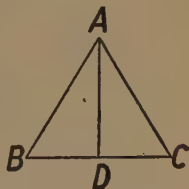
right triangle if each of the equal sides is 1 in. ; 2 in. ; 3 in. ; a inches.

6. Find the hypotenuse of a right triangle if the other sides are 6 in. and 10 in.

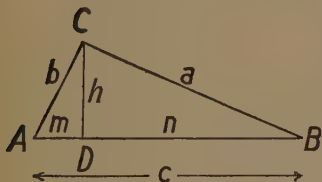
7. If the hypotenuse of a right triangle is 15 in. and one of the other sides is 10 in., what is the third side?

The altitude to the base of an isosceles triangle bisects the base of the triangle.

8. Find the altitude, AD , if $AB = 10$ and $BC = 8$.



9. Find AB if $AD = 20$ and $CD = 12$.



In the figure at the left, h is the altitude to the hypotenuse of the right triangle ABC , $h^2 = mn$, $b^2 = cm$, and $a^2 = cn$ (see ex. 6, page 228).

10. Find CD if $m = 8$ and $n = 6$.

11. Find AC if $c = 15$ and $m = 3$.

12. Find CB if $c = 18$ and $n = 12$.

13. If $a = 15$, $b = 12$ and $m = 9$, find (a) CD ; (b) DB .

14. Find the radius of a circle if the area is 396 sq. in. ($A = \pi r^2$).

15. Find one side of a square if its area is 675 sq. ft.

Rationalizing the Denominators of Fractions

If you are required to find the value of $\frac{2}{\sqrt{3}}$, you may first get the value of $\sqrt{3}$ from the table on page 227 and then divide 2 by this value.

$$\text{Thus, } \frac{2}{\sqrt{3}} = \frac{2}{1.732} = 1.154 \dots$$

The work will be simpler, however, if you first make the denominator of the fraction a rational number. This may be done by *multiplying both terms of the fraction by the smallest number that will make the denominator a rational number.*

$$\text{Thus, } \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2}{3} \sqrt{3}. \quad \text{Since } \sqrt{3} = 1.732, \frac{2}{3}\sqrt{3} = 1.154$$

The factor $\sqrt{3}$ is called the **rationalizing factor** of the denominator.

EXAMPLE 1. Express $\sqrt{\frac{3a^2}{8b^3}}$ with a rational denominator.

$$\text{SOLUTION.} \quad \sqrt{\frac{3a^2}{8b^3}} = \sqrt{\frac{3a^2(2b)}{8b^3(2b)}} = \sqrt{\frac{6a^2b}{16b^4}} = \frac{a}{4b^2} \sqrt{6b}$$

EXAMPLE 2. Find the value of $\frac{3}{\sqrt[3]{2}}$.

SOLUTION. The smallest factor that will make the denominator rational is $\sqrt[3]{4}$.

$$\frac{3}{\sqrt[3]{2}} = \frac{3\sqrt[3]{4}}{\sqrt[3]{2} \cdot \sqrt[3]{4}} = \frac{3\sqrt[3]{4}}{\sqrt[3]{8}} = \frac{3}{2} \sqrt[3]{4} = \frac{3}{2} \times 1.587 = 2.38$$

EXAMPLE 3. Express $\frac{2}{\sqrt[3]{a^2}}$ with a rational denominator.

$$\text{SOLUTION.} \quad \frac{2}{\sqrt[3]{a^2}} = \frac{2\sqrt[3]{a}}{\sqrt[3]{a^2} \cdot \sqrt[3]{a}} = \frac{2\sqrt[3]{a}}{\sqrt[3]{a^3}} = \frac{2\sqrt[3]{a}}{a}$$

Note that the resulting fraction must be expressed in its simplest form.

You will notice that rationalization of the denominator of fractions involves *division by a radical.*

EXERCISES

Express each of the following with a rational denominator :

- | | | |
|---------------------------------|-------------------------------------|--|
| 1. $\frac{1}{\sqrt{3}}$ | 11. $\sqrt{6\frac{1}{8}}$ | 21. $\frac{1}{\sqrt[3]{4}}$ |
| 2. $\frac{1}{\sqrt{2}}$ | 12. $\sqrt{4\frac{1}{8}}$ | 22. $\frac{6}{\sqrt[3]{2}}$ |
| 3. $\sqrt{\frac{1}{2}}$ | 13. $\sqrt{5\frac{3}{5}}$ | 23. $\frac{12}{\sqrt[3]{6}}$ |
| 4. $\frac{2}{\sqrt{3}}$ | 14. $\frac{a\sqrt{b}}{\sqrt{a}}$ | 24. $\frac{8}{\sqrt[3]{12}}$ |
| 5. $\sqrt{\frac{2}{3}}$ | 15. $\sqrt{\frac{A}{\pi}}$ | 25. $\frac{6}{\sqrt[3]{18}}$ |
| 6. $\sqrt{\frac{3}{5}}$ | 16. $\frac{\sqrt{ab}}{\sqrt{a}}$ | 26. $\frac{a}{\sqrt[3]{18}}$ |
| 7. $\frac{\sqrt{2}}{\sqrt{5}}$ | 17. $\frac{2\sqrt{a}}{\sqrt{3}b}$ | 27. $\frac{\sqrt[3]{4}}{\sqrt[3]{6}}$ |
| 8. $\frac{\sqrt{12}}{\sqrt{3}}$ | 18. $\frac{\sqrt{14}a}{\sqrt{7}}$ | 28. $\frac{\sqrt[3]{9}}{\sqrt[3]{12}}$ |
| 9. $\frac{\sqrt{15}}{\sqrt{5}}$ | 19. $\frac{7\sqrt{48}x}{\sqrt{12}}$ | 29. $\frac{a}{\sqrt[3]{a^2}}$ |
| 10. $\frac{3}{\sqrt{5}}$ | 20. $\sqrt{\frac{8a}{3b}}$ | 30. $\frac{a^2}{\sqrt[3]{4a^2}}$ |

Find the value to the nearest tenth :

- | | | |
|---------------------------------|----------------------------------|---------------------------------------|
| 31. $\frac{3}{\sqrt{2}}$ | 34. $\frac{2\sqrt{3}}{\sqrt{2}}$ | 37. $\frac{5}{\sqrt[3]{2}}$ |
| 32. $\frac{2}{\sqrt{5}}$ | 35. $\sqrt{\frac{18}{5}}$ | 38. $\frac{3}{\sqrt[3]{3}}$ |
| 33. $\frac{\sqrt{2}}{\sqrt{5}}$ | 36. $\sqrt{\frac{2}{\pi}}$ | 39. $\frac{\sqrt[3]{3}}{\sqrt[3]{4}}$ |

Adding and Subtracting Radicals

Before radicals can be combined by addition and subtraction, they must have the same index and the same radicand. Such radicals are called **like radicals** or **similar radicals**.

Thus, $3a\sqrt{2}$ and $5b\sqrt{2}$ are like radicals and may be combined to read $(3a + 5b)\sqrt{2}$.

EXAMPLE. Combine the radicals $\sqrt{12} + 3\sqrt{\frac{4}{3}} - \frac{2}{3}\sqrt{75} + \sqrt{\frac{49}{3}}$.

SOLUTION. First simplify each radical, then combine like radicals.

$$\sqrt{12} + 3\sqrt{\frac{4}{3}} - \frac{2}{3}\sqrt{75} + \sqrt{\frac{49}{3}} = 2\sqrt{3} + 2\sqrt{3} - \frac{10}{3}\sqrt{3} + \frac{7}{3}\sqrt{3} = 3\sqrt{3}$$

EXERCISES

Simplify, and combine like radicals:

1. $2\sqrt{2} - 3\sqrt{32} + 4\sqrt{18} - \sqrt{8}$
2. $5\sqrt{5} - 2\sqrt{80} + \sqrt{245} - 3\sqrt{45}$
3. $2\sqrt{3} - 3\sqrt{27} + \frac{1}{2}\sqrt{75} - \frac{1}{3}\sqrt{12}$
4. $3\sqrt{8} - 2\sqrt{18} + \sqrt{50} + 2\sqrt{32}$
5. $5\sqrt{\frac{1}{2}} - 2\sqrt{\frac{1}{8}} + \sqrt{8} - \sqrt{18}$
6. $2\sqrt{150} - 4\sqrt{54} + 6\sqrt{24} + \sqrt{96}$
7. $2\sqrt{54} - 6\sqrt{\frac{2}{3}} - \sqrt{96} + \sqrt{\frac{3}{2}}$
8. $2\sqrt{50} - 3\sqrt{2} + 5\sqrt{72} - 7\sqrt{8}$
9. $\sqrt{3} - 2\sqrt{27} + 3\sqrt{12} + 3\sqrt{\frac{4}{3}}$
10. $5\sqrt{11} - 3\sqrt{44} + \sqrt{99} - 22\sqrt{\frac{9}{11}}$
11. $4\sqrt{\frac{1}{2}} + \sqrt{12\frac{1}{2}} - \sqrt{\frac{1}{8}} - 3\sqrt{1\frac{1}{8}}$
12. $3\sqrt{5x^2} - 10x\sqrt{\frac{1}{5}} + 2\sqrt{45x^2} - 5x\sqrt{9\frac{4}{5}}$
13. $4\sqrt{27} - 3\sqrt{12} + 5\sqrt{\frac{4}{3}} - \sqrt{\frac{25}{3}}$
14. $60\sqrt{\frac{1}{5}} - 8\sqrt{\frac{9}{20}} + 17\sqrt{\frac{1}{5}} - 23\sqrt{500}$

Multiplying Radicals

The method of multiplying radicals of the same order is shown in the following examples.

EXAMPLE 1. Find the product of $2\sqrt{3} \times \sqrt{12} \times 3\sqrt{75}$.

$$\begin{aligned}\text{SOLUTION. } 2\sqrt{3} \times \sqrt{12} \times 3\sqrt{75} &= 6\sqrt{3 \times 12 \times 75} \\ &= 6\sqrt{3^2 \times 2^2 \times 5^2 \times 3} = 180\sqrt{3}\end{aligned}$$

EXAMPLE 2. Find the product of $5\sqrt[3]{4ax^2}$ by $2\sqrt[3]{6a^2x^2}$.

$$\begin{aligned}\text{SOLUTION. } 5\sqrt[3]{4ax^2} \cdot 2\sqrt[3]{6a^2x^2} &= 5 \times 2\sqrt[3]{4ax^2 \cdot 6a^2x^2} \\ &= 10\sqrt[3]{8a^3x^3(3x)} = 20ax\sqrt[3]{3x}\end{aligned}$$

Always express the result in the simplest form.

EXERCISES

Perform the indicated multiplication. Do ex. 1 to 15 orally :

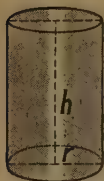
- | | | |
|--|--|------------------------------|
| 1. $\sqrt{2} \cdot \sqrt{8}$ | 6. $3\sqrt{3} \cdot 5\sqrt{15}$ | 11. $(3\sqrt{2})(3\sqrt{2})$ |
| 2. $\sqrt{3} \cdot \sqrt{12}$ | 7. $2\sqrt{6} \cdot \sqrt{12}$ | 12. $(5\sqrt{3})^2$ |
| 3. $\sqrt{18} \cdot \sqrt{2}$ | 8. $3\sqrt{14} \cdot \sqrt{7}$ | 13. $(2a\sqrt{3a})^2$ |
| 4. $\sqrt{5} \cdot \sqrt{15}$ | 9. $4\sqrt{20} \cdot \sqrt{2}$ | 14. $(3\sqrt{2ax})^2$ |
| 5. $2\sqrt{8} \cdot \sqrt{6}$ | 10. $7\sqrt{2} \cdot \sqrt{32}$ | 15. $(5xy\sqrt{2x})^2$ |
| 16. $\sqrt{15}(\sqrt{3} - \sqrt{15} + \sqrt{10})$ | 18. $5\sqrt{5}(3\sqrt{5} - \sqrt{20} + \sqrt{10})$ | |
| 17. $\sqrt{5}(\sqrt{15} + 2\sqrt{3} - \sqrt{10})$ | 19. $3\sqrt{2}(\sqrt{6} - \sqrt{3} + 5\sqrt{8})$ | |
| 20. $(3\sqrt{2} - 2\sqrt{3})(5\sqrt{2} + 4\sqrt{3})$ | | |

$$\begin{array}{r} \text{SOLUTION. } 3\sqrt{2} - 2\sqrt{3} \\ 5\sqrt{2} + 4\sqrt{3} \\ \hline 30 - 10\sqrt{6} \\ + 12\sqrt{6} - 24 \\ \hline 30 + 2\sqrt{6} - 24 = 6 + 2\sqrt{6} \end{array}$$

- | | |
|--------------------------------------|--------------------------------------|
| 21. $(5\sqrt{2} - 3)(5\sqrt{2} + 3)$ | 24. $(2\sqrt{3} - 1)^2$ |
| 22. $(3\sqrt{3} - 1)(3\sqrt{3} + 5)$ | 25. $(3\sqrt{5} - \sqrt{2})^2$ |
| 23. $(2\sqrt{5} - 2)(3\sqrt{5} + 2)$ | 26. $(2\sqrt{3} + 1)(2\sqrt{3} - 1)$ |

For *division by a radical*, see page 236.

EXERCISES ON GEOMETRIC FORMULAS



1. The formula for the volume of a cylinder is $V = \pi r^2 h$. (a) Solve for r in terms of the other letters.

(b) Find the radius of a cylinder if $V = 396$ cu. in., $h = 10$ in., and $\pi = 3\frac{1}{7}$.



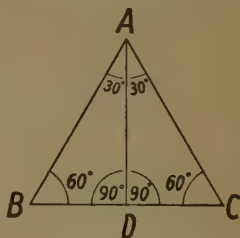
2. The formula for the volume of a cone is $V = \frac{1}{3} \pi r^2 h$. (a) Solve for r in terms of the other letters.

(b) Find r when $V = 100$, $h = 7$, and $\pi = 3\frac{1}{7}$.

3. The formula for the surface of a sphere is $S = 4 \pi r^2$. (a) Solve this formula for r .

(b) Find r in terms of π if $S = 250$ sq. in.

The altitude of an equilateral triangle bisects the vertex angle and divides the equilateral triangle into two equal right triangles. Since each angle of the equilateral triangle is 60° , the angles of each of the right triangles are 90° , 60° , and 30° . The side BD opposite the 30° angle is $\frac{1}{2} BC$. Hence it is $\frac{1}{2} AB$ and $\frac{1}{2} AC$. Such a right triangle is called a 30° - 60° triangle and is an important one in geometry.

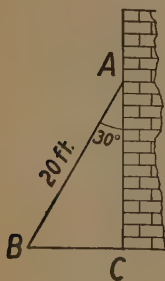


4. Find DC and AD if $AC = 5$ in.

5. Find, to the nearest tenth, the altitude of an equilateral triangle if one side is 10 in.

6. In a 30° - 60° right triangle, find the other two sides if the side opposite the 60° angle is 5 in.; h inches.

7. A ladder 20 ft. long is raised against a house so that it makes an angle of 30° with the side of the house. How far from the house is the foot of the ladder? How high above the ground is the top of the ladder?



Exponents

You are familiar with the following general laws of exponents:

Multiplication. $a^m \cdot a^n = a^{m+n}$

Division. $a^m \div a^n = a^{m-n}$

Powers. $(a^m)^n = a^{mn}$

These laws are assumed to be true for any values of m and n , whether integral or fractional.

Thus, $a^m \cdot a^n = (a \cdot a \cdot a \dots \text{to } m \text{ factors})(a \cdot a \cdot a \dots \text{to } n \text{ factors})$
 $= a^{m+n}$

$$\frac{a^m}{a^n} = \frac{a \cdot a \cdot a \dots \text{to } m \text{ factors}}{a \cdot a \cdot a \dots \text{to } n \text{ factors}}$$

$$= [a \cdot a \cdot a \dots \text{to } (m - n) \text{ factors}] = a^{m-n}$$

$$(a^m)^n = a^m \cdot a^m \cdot a^m \dots \text{to } n \text{ factors} = a^{mn}$$

The meaning of **fractional exponents**, which obey the same laws as integral exponents, is made clear in the following examples.

EXAMPLE 1. Find the product of $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}}$.

SOLUTION. According to the law of exponents:

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2} + \frac{1}{2}} = a^{\frac{2}{2}}, \text{ or } a$$

That is, $a^{\frac{1}{2}}$ is one of the two equal factors of a , or $a^{\frac{1}{2}} = \sqrt{a}$.

EXAMPLE 2. Find the product of $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}$.

SOLUTION. $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = a^{\frac{3}{3}} = a$

That is, $a^{\frac{1}{3}}$ is one of the three equal factors of a . Hence $a^{\frac{1}{3}} = \sqrt[3]{a}$.

EXAMPLE 3. Find the product of $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}}$.

SOLUTION. $a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3} + \frac{1}{3}} = a^{\frac{2}{3}}, \text{ or } \sqrt[3]{a} \cdot \sqrt[3]{a} = \sqrt[3]{a^2}$

From these examples, it is seen that *the denominator of a fractional exponent indicates a root and the numerator indicates a power.*

Thus, $5^{\frac{3}{2}}$ indicates the square root of the cube of 5, or $5^{\frac{3}{2}} = \sqrt{5^3}$, and $a^{\frac{n}{m}}$ indicates the m th root of the n th power of a , or $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.

Read $a^{\frac{2}{3}}$ as “ a exponent two thirds” or “the cube root of the square of a .” Do not say “ a raised to the two thirds power,” as this has no meaning.

EXERCISES

State the value of :

- | | | | |
|-----------------------|-----------------------|-----------------------|-------------------------|
| 1. $4^{\frac{1}{2}}$ | 4. $8^{\frac{1}{3}}$ | 7. $4^{\frac{2}{3}}$ | 10. $144^{\frac{3}{4}}$ |
| 2. $9^{\frac{1}{2}}$ | 5. $27^{\frac{1}{3}}$ | 8. $8^{\frac{2}{3}}$ | 11. $32^{\frac{3}{5}}$ |
| 3. $16^{\frac{1}{2}}$ | 6. $64^{\frac{1}{3}}$ | 9. $16^{\frac{3}{4}}$ | 12. $64^{\frac{3}{4}}$ |

Write the following with radical signs :

- | | | | |
|-----------------------|--------------------------|--------------------------|--------------------------------------|
| 13. $a^{\frac{1}{2}}$ | 15. $(2a)^{\frac{1}{2}}$ | 17. $(3a)^{\frac{1}{3}}$ | 19. $x^{\frac{1}{2}}y^{\frac{2}{3}}$ |
| 14. $a^{\frac{1}{3}}$ | 16. $2a^{\frac{1}{2}}$ | 18. $(5a)^{\frac{2}{3}}$ | 20. $a^{\frac{1}{2}}b^{\frac{3}{4}}$ |

Express the product of the following with exponents :

21. $x^{\frac{1}{3}} \cdot x^{\frac{1}{2}}$ SOLUTION. $x^{\frac{1}{3}} \cdot x^{\frac{1}{2}} = x^{\frac{1}{3} + \frac{1}{2}} = x^{\frac{5}{6}}$
22. $a^{\frac{1}{2}} \cdot a^{\frac{2}{3}}$ 24. $3x^{\frac{2}{3}} \cdot x^{\frac{3}{2}}$ 26. $2a^{\frac{1}{4}} \cdot a^{\frac{1}{2}}$ 28. $(4a)^{\frac{3}{4}}a^{\frac{1}{2}}$
23. $2x^{\frac{3}{2}} \cdot x^{\frac{1}{2}}$ 25. $(5x)^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$ 27. $3a \cdot a^{\frac{1}{3}}$ 29. $(8a)^{\frac{2}{3}}a^{\frac{1}{3}}$

30-37. Express the products in ex. 22 to 29 in simplified radical form.

Express each of the following with fractional exponents and find the indicated products :

- | | | |
|------------------------------------|------------------------------------|-------------------------------------|
| 38. $\sqrt{2} \cdot \sqrt[3]{2}$ | 41. $\sqrt[3]{5} \cdot \sqrt{5}$ | 44. $\sqrt[4]{a^3} \cdot \sqrt{a}$ |
| 39. $\sqrt[3]{2^2} \cdot \sqrt{2}$ | 42. $\sqrt{a} \cdot \sqrt[3]{a}$ | 45. $\sqrt[4]{a} \cdot \sqrt[3]{a}$ |
| 40. $\sqrt{3} \cdot \sqrt[3]{3^2}$ | 43. $\sqrt[3]{a^2} \cdot \sqrt{a}$ | 46. $\sqrt{27} \cdot \sqrt[3]{3}$ |

47-55. Express the products in ex. 38 to 46 in simplified radical form.

Any number with a zero exponent is equal to 1. (See page 68.)

Thus, $a^m \div a^m = a^{m-m} = a^0$; but $a^m \div a^m = 1$. Therefore $a^0 = 1$.

Radicals may sometimes be simplified by reducing the index of the root.

EXAMPLE. Simplify $\sqrt[4]{4} + \sqrt[6]{8} + \sqrt[6]{9}$.

SOLUTION. $\sqrt[4]{4} + \sqrt[6]{8} + \sqrt[6]{9} = \sqrt[2]{2^2} + \sqrt[3]{2^3} + \sqrt[3]{3^2} = 2^{\frac{2}{2}} + 2^{\frac{3}{3}} + 3^{\frac{2}{3}}$
 $= 2^{\frac{1}{1}} + 2^{\frac{1}{1}} + 3^{\frac{1}{3}} = \sqrt{2} + \sqrt{2} + \sqrt[3]{3} = 2\sqrt{2} + \sqrt[3]{3}$

Express each radicand as a power of some number, then write the radical with fractional exponents reduced to lowest terms. Express the result with radical signs.

EXERCISES

State the value of :

- | | | | |
|------------------------|-------------------------------|-----------------------|-----------------------|
| 1. $121^{\frac{1}{2}}$ | 3. $3 \times 3^{\frac{1}{2}}$ | 5. 25^0 | 7. $64^{\frac{3}{2}}$ |
| 2. 2^0 | 4. $-2 \cdot 2^{\frac{3}{2}}$ | 6. $25^{\frac{1}{2}}$ | 8. $64^{\frac{2}{3}}$ |

Express each of the following with smaller root index :

- | | | |
|-------------------------|------------------------|----------------------------|
| 9. $\sqrt[4]{9}$ | 12. $\sqrt[8]{16 x^4}$ | 15. $\sqrt[4]{a^6 x^{10}}$ |
| 10. $\sqrt[6]{27}$ | 13. $\sqrt[6]{16 a^4}$ | 16. $\sqrt[6]{a^9 x^3}$ |
| 11. $\sqrt[4]{a^2 x^6}$ | 14. $\sqrt[4]{25 x^2}$ | 17. $\sqrt[4]{64 x^6 y^8}$ |

Find the indicated quotients :

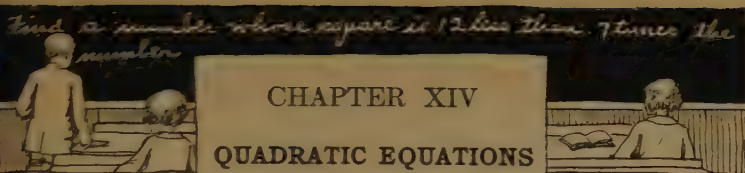
- | | | |
|---|---|---|
| 18. $\frac{x^{\frac{3}{2}}}{x^{\frac{4}{3}}}$ | SOLUTION. $\frac{x^{\frac{3}{2}}}{x^{\frac{4}{3}}} = x^{\frac{3}{2} - \frac{4}{3}} = x^{\frac{1}{6}}$ | |
| 19. $\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$ | 21. $\frac{x^{\frac{5}{2}}}{x}$ | 23. $\frac{a^{\frac{1}{2}}}{a^{\frac{1}{4}}}$ |
| 20. $\frac{a^3}{a^{\frac{2}{3}}}$ | 22. $\frac{2 x^{\frac{1}{2}}}{3 x^{\frac{1}{3}}}$ | 24. $\frac{a}{a^{\frac{2}{3}}}$ |

Find the indicated powers :

- | | | |
|---------------------------|--|--|
| 25. $(\sqrt[3]{a^2})^2$ | SOLUTION. $(\sqrt[3]{a^2})^2 = (a^{\frac{2}{3}})^2 = a^{\frac{4}{3}} = a\sqrt[3]{a}$ | |
| 26. $(x^3)^2$ | 31. $(\sqrt{x^3})^2$ | 36. $\left(\frac{2\sqrt{3}}{\sqrt{5}}\right)^3$ |
| 27. $(x^2)^3$ | 32. $(\sqrt{5x})^0$ | 37. $\left(\frac{7\sqrt[3]{4x^2}}{2}\right)^2$ |
| 28. $(x^{\frac{3}{2}})^4$ | 33. $(3\sqrt{8x})^3$ | 38. $\left(\frac{\sqrt{2x}}{\sqrt[3]{4x^2}}\right)^3$ |
| 29. $(\sqrt[3]{x})^4$ | 34. $(8^0 \cdot 125^{\frac{1}{3}})^2$ | 39. $\left(\frac{\sqrt{6} - \sqrt{15}}{\sqrt{3}}\right)^2$ |
| 30. $(\sqrt[4]{x^2})^3$ | 35. $(64^{\frac{2}{3}})^{\frac{3}{4}}$ | |

REVIEW EXERCISES

1. The order of a radical is determined by —.
2. A radical is expressed in simplest form when —.
3. The denominator of a fraction may be rationalized by —.
4. A rationalizing factor is —.
5. The law of exponents for multiplication is —.
6. The law of exponents for division is —.
7. The law of exponents for raising to a power is —.
8. The numerator of a fractional exponent indicates — and the denominator indicates —.
9. Any number having a zero exponent is —.
10. The theorem of Pythagoras states that —.
11. In a 30° - 60° right triangle the side opposite the 30° angle is —.
12. The formula for the surface of a sphere is —.
13. The formula for the volume of a cylinder is —.
14. The formula for the volume of a cone is —.
15. The formula for the volume of a sphere is $V = \frac{4}{3} \pi r^3$.
- (a) Solve this formula for r .
- (b) Find r in terms of π if $V = 288$ cu. in.
16. The formula $S = \frac{1}{2} g t^2$ gives the distance a freely falling body will drop in t seconds. (a) Solve the formula for t .
- (b) Find t , to the nearest tenth, if $g = 32$ and $S = 100$.
17. (a) Solve the formula $h = \frac{v^2}{2g}$ for v .
- (b) Find the value of v when $h = 25$ and $g = 32.16$.
18. Define rational and irrational radicals. Explain how an irrational quantity may be represented graphically.
19. What is meant by the principal square root of a number? What is the value of $\sqrt{9} - \sqrt{4}$? If the principal roots are not considered, what values may be given to $\sqrt{9} - \sqrt{4}$?
20. Are $x = 2\sqrt{5}$ and $y = 3\sqrt{5}$ roots of the equation $3x^2 - 4xy + y^2 = 65$? of $3x - 2y = 0$?



Incomplete Quadratic Equations

As you continue your work in mathematics, you will find that many problems involve equations which may appear quite simple, but which cannot be solved by the methods that you have already learned.

Suppose, for example, that you wish to solve this problem: What is the side of a square if its area increased by 4 sq. in. equals 29 sq. in.?

You may write the equation:

$$n^2 + 4 = 29$$

The solution of this equation differs from those which you have studied because it contains the term n^2 . You cannot solve this equation without learning some new principles.

SOLUTION.

$$n^2 + 4 = 29$$

Subtract 4 from each side:

$$n^2 = 25$$

Since the value of n is required,

find the square root of each side: $n = +5$ or -5 *Ans. 5 in.*

NOTE. Since $(+5)^2 = 25$ and $(-5)^2 = 25$, n has two values which may be written as $n = \pm 5$. While the negative result is a root of the equation, it has no meaning in this problem, since a square cannot be a negative number of inches long. In such cases the negative results should be dropped.

In solving the equation $n^2 + 4 = 29$, you used the new principle that *equal roots of equal numbers are equal*.

An equation that in its simplest integral form contains the second power of the unknown number, but no higher powers, is called a **quadratic equation**.

The simplest quadratic equations are those in which the first power of the variable is lacking, as $ax^2 + c = 0$. Such equations are called **incomplete quadratic equations**.

EXAMPLE. Solve the equation $(3x - 7)(3x + 7) = 15$.

SOLUTION.

$$(3x - 7)(3x + 7) = 15$$

Find the product :

$$9x^2 - 49 = 15$$

$$9x^2 = 64$$

Find the square root :

$$3x = 8 \text{ or } -8$$

$$x = \frac{8}{3} \text{ or } -\frac{8}{3}$$

The result is frequently written thus: $x = \pm \frac{8}{3}$

Check the solution by substituting each root in the original equation.

EXERCISES

Solve and check :

1. $5x^2 - 5 = 120$
2. $8x^2 = 2x^2 + 384$
3. $3x^2 - 28 = 215$
4. $9x^2 + 5 = 174$
5. $3x^2 - 8 = x^2 + 120$
6. $(x - 6)(x + 6) = 13$
7. $(2x - 7)(2x + 7) = 15$
8. $(3x + 5)(3x - 5) = 56$
9. $\frac{x - 3}{4} = \frac{4}{x + 3}$
10. $\frac{4 + x}{4 - x} = \frac{x + 9}{x - 9}$
11. $(x + \frac{1}{2})(x - \frac{1}{2}) = 6$
12. $(x + \frac{1}{3})(x - \frac{1}{3}) = 2\frac{2}{3}$
13. $(2x - 5)^2 = 2x(x - 10) + 169$
14. $(2x - 4)(3x + 9) = 6(x + 10)$
15. $(x + 2)^3 - (x - 2)^3 = 124$
16. $(2x - 7)^2 = 2(29 - 14x)$
17. $(2x + 11)^2 = 2(22x + 73)$
18. $(x + 3)(x - 5) = 2(17 - x)$
19. $\frac{5x - 15}{2x + 1} = \frac{x - 3}{x + 2}$
20. $\frac{x + 4}{x + 16} = \frac{4 - x}{x - 16}$
21. Two numbers are to each other as 3 to 5 and their product is 1215. What are the numbers?
22. Find a number that multiplied by $\frac{1}{4}$ of itself equals 16.
23. A rectangular field is $\frac{1}{3}$ as wide as it is long and contains $7\frac{1}{2}$ A. How many rods long is it? (An acre is 160 sq. rd.)
24. Find the dimensions of a rectangular building lot that contains 4725 sq. ft. if the sides are to each other as 3 to 7.
25. Find the length of a diagonal path across a square field that contains 4 A.
26. The area of a triangular field is 1728 sq. rd. Find the altitude and the base line if they are as 2 to 3.

Complete Quadratic Equations

The equations that you have worked on page 246 are, when simplified, of the form $ax^2 + c = 0$. These have been easily solved. However, if the equation contains a term in x , it takes the form $ax^2 + bx + c = 0$. Such an equation is called a **complete quadratic equation**. Its solution requires new methods.

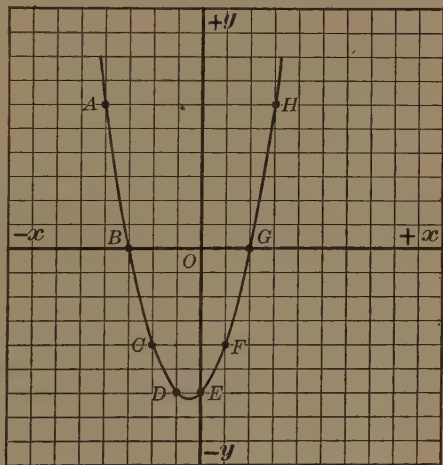
EXAMPLE. Jane wishes to make a rug that will be 1 ft. longer than it is wide and that will have an area of 6 sq. ft. What must be the dimensions of the rug?

This problem gives rise to the equation $x(x + 1) = 6$, or $x^2 + x - 6 = 0$. It is so simple a problem that you have probably already guessed the answer, but guessing will not solve most of your problems.

Graphic solution of quadratic equations. A graph of the equation $x^2 + x - 6 = 0$ may be constructed if the equation is written

$$y = x^2 + x - 6$$

From this equation construct a table of values for x and y that satisfy the equation.



	A	B	C	D	E	F	G	H	etc.
$x =$	-4	-3	-2	-1	0	1	2	3	etc.
$y =$	6	0	-4	-6	-6	-4	0	6	etc.

These pairs of values have been plotted on the graph above at points A, B, C, etc., and a smooth curve has been drawn

through the points. This line is the graph of the function $y = x^2 + x - 6$. B and G represent the points at which $y = 0$. They represent the roots of the equation $x^2 + x - 6 = 0$.

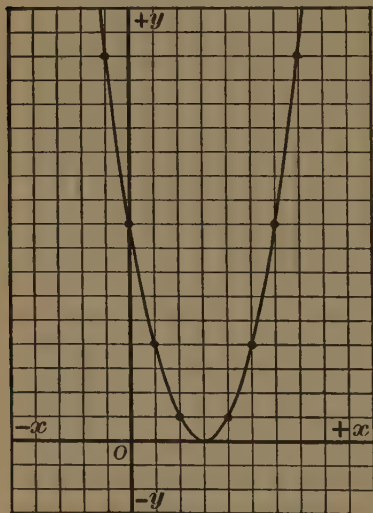
Notice that the values of y in the graph on page 247 are determined by the corresponding values of x . The curve is the locus of all points that satisfy the condition $y = x^2 + x - 6$. The graphs of equations that have this form are curved lines similar to that below. Such curves are called **parabolas**.

EXAMPLE. Draw a graph of the equation $x^2 - 6x + 9 = 0$.

SOLUTION. First write the equation as $y = x^2 - 6x + 9$. Construct a table of values of x and y that satisfy the equation.

$x =$	-1	0	1	2	3	4	5	6	7	etc.
$y =$	16	9	4	1	0	1	4	9	16	etc.

The pairs of values, $x = -1, y = 16$; $x = 0, y = 9$, etc., are plotted as shown in the graph below and a smooth curve is drawn through the points.



Unlike the circle, which is a closed curve, the ends of this curve will never come together.

If $y = 0$, the equation becomes $x^2 - 6x + 9 = 0$. The roots of this equation are the points where the line $y = 0$ (the x -axis) crosses the curve. In this case the x -axis is tangent to the curve at the point $x = 3$.

The graph on page 247 indicates that in general there will be two roots to a quadratic equation. The two roots of $y = x^2 - 6x + 9$ are identical, each being 3.

EXERCISES

1. If $y = 4$, the equation $y = x^2 - 6x + 9$ becomes $4 = x^2 - 6x + 9$, or $x^2 - 6x + 5 = 0$. Place your ruler along the horizontal line that indicates the line $y = 4$ in the graph on page 248. What are the points in which the curve cuts the line $y = 4$? What, then, are the roots of the equation $x^2 - 6x + 5 = 0$? Substitute these roots in the equation. Do they satisfy the equation?

2. If $y = 2$, the equation $y = x^2 - 6x + 9$ becomes $x^2 - 6x + 7 = 0$. At what points does the line $y = 2$ cut the curve? What, then, are the approximate roots of the equation $x^2 - 6x + 7 = 0$?

3. If $y = -2$, the equation $x^2 - 6x + 9$ becomes $x^2 - 6x + 11 = 0$. What relation does the line $y = -2$ bear to the graph? The graph does not show the roots of this equation. The roots are said to be **imaginary**. Such values are studied in more advanced algebra.

Construct graphs of the following equations. From the graphs determine the roots of the equations. Check the roots by substitution:

4. $x^2 - 1 = 0$

9. $x^2 + x - 12 = 0$

5. $x^2 - 16 = 0$

10. $x^2 + 7x + 12 = 0$

6. $x^2 + x - 2 = 0$

11. $x^2 - 7x + 12 = 0$

7. $x^2 - x - 2 = 0$

12. $x^2 - 7x + 6 = 0$

8. $x^2 - 3x + 2 = 0$

13. $x^2 + 5x - 6 = 0$

14. Construct the graph of $12x^2 + 23x - 9 = y$. From the graph determine the approximate roots of the equation when $y = 0$; when $y = 7$.

15. Construct the graph of $x^2 - 6x + 10 = 0$. What does the graph tell you about the roots of the equation?

16. How many points is it desirable to plot before constructing the graph of a quadratic equation?

Equations That Can Be Solved by Factoring

You have seen that quadratic equations may be solved graphically; but this method of solution has several disadvantages. It is therefore well to be familiar with other methods of solving such equations.

The following example shows how factoring may be used in the solution of some quadratic equations.

EXAMPLE. Solve the equation $x^2 - 7x = 18$.

SOLUTION. $x^2 - 7x = 18$ (1)

Subtract 18 from each side: $x^2 - 7x - 18 = 0$ (2)

Factor the left side of (2): $(x - 9)(x + 2) = 0$ (3)

It is evident that if the product of two numbers is equal to zero, either number must be zero or both numbers must be zero. Thus, if $ab = 0$, then $a = 0$ or $b = 0$ or both a and b equal zero. Accordingly, from (3) above, either $x - 9 = 0$ or $x + 2 = 0$ or both factors equal 0.

Hence $x = 9$ or $x = -2$.

CHECK. Check by substitution.

If $x = 9$, $x^2 - 7x = 81 - 63 = 18$

If $x = -2$, $x^2 - 7x = 4 + 14 = 18$ Hence the values are correct.

If the product of two factors is zero, then at least one of the factors is zero.

Instead of substituting the values of x in the original equation, the checking of equations of the form $x^2 + bx + c = 0$, when the coefficient of x^2 is 1, may be done in a simpler way. Notice that the sum of the two roots is $9 - 2$, or 7, which is the coefficient of x with its sign changed, and that the product of the two roots is 18, the known term. Hence the roots of the equation are 9 and -2 . Compare this with the method of factoring $x^2 + bx + c$ (page 162).

In general, if the roots of the equation $x^2 + bx + c = 0$ are correct, their product equals the known term, c , and their sum equals the coefficient of x with its sign changed. If the equation is of the form $ax^2 + bx + c = 0$, it should be changed before checking to the form $x^2 + \frac{bx}{a} + \frac{c}{a} = 0$ by dividing by a .

EXERCISES

Solve the equations and check :

1. $x^2 - 5x = 6$

2. $x^2 - 8x = 9$

3. $x^2 + x = 12$

4. $x^2 + 2x = 15$

5. $x^2 - 3x = 10$

6. $x^2 + 7x = -10$

7. $x^2 + 5 = 6x$

8. $x^2 + 12 = 8x$

9. $3x^2 + 6x = 24$

HINT. Simplify by dividing by 3.

10. $2x^2 = 14x + 60$

11. $x^2 - 3x = 180$

12. $x^2 + 2x = 143$

13. $6x^2 + 1 = 5x$

14. $2x^2 - 7x + 3 = 0$

15. $6x^2 + x = 2$

16. $6x^2 + 21x = 45$

17. $2x^2 + 6 = 7x$

18. $3x^2 + 7 = 10x$

19. $5x^2 + 14x = 3$

20. $x^2 - 2ax + a^2 = 0$

21. $x^2 + 3a^2 = 4ax$

22. $x^2 - ax + 2a = 2x$

23. $x^2 + ab = ax + bx$

24. $x^2 = 3bx - 2b^2$

25. $21x^2 = 2dx + 3d^2$

26. A rectangular lot is twice as long as a square lot and its width is 5 rd. less. If the area of the rectangular lot is 132 sq. rd., find the dimensions of each lot.

SOLUTION. Let s = the number of rods in length of the square lot

Then $2s$ = the number of rods in length of the rectangular lot

and $s - 5$ = the number of rods in width of rectangular lot

Then $2s(s - 5) = 132$, or $s^2 - 5s = 66$

Solve the equation: $s = 11$ or -6

Since the side of the square cannot be a negative quantity, disregard -6 . Then the dimensions of the rectangular lot are 22 rd. and 6 rd.

27. A rectangular lot is 3 times as long as a square lot and 10 ft. wider. The area of the rectangular lot is 18,900 sq. ft. greater than the area of the square lot. Find the dimensions of each lot.

28. The length of a field exceeds its breadth by 24 rd. and its area is 4 A. What are the dimensions of the lot?

29. A rug is 4 ft. longer than it is wide and has an area of 21 sq. ft. Find the dimensions of the rug.

Completing the Square

You will find that many quadratic equations cannot be solved by factoring, as the algebraic expressions have no rational factors.

For example, the left side of the equation $x^2 + 4x + 1 = 0$ cannot be factored. To solve such an equation it is necessary to find a way to make the left side of the equation a square.

The square of a binomial is equal to the square of the first term plus or minus twice the product of the first term by the second term plus the square of the second term. (See page 169.)

Thus, $(x \pm a)^2 = x^2 \pm 2ax + a^2$.

An expression like $x^2 \pm 2ax$ can be made a square by adding to the expression the square of one half the coefficient of x .

Thus, $x^2 \pm 2ax + \left(\frac{2a}{2}\right)^2 = x^2 \pm 2ax + a^2$.

EXAMPLE. Add a term to $x^2 + 6ax$ to make it a square.

SOLUTION. The first term is a square, x^2 . The second term is twice the product of the square root of the first term, x , by $3a$, or $2x(3a)$. The term that is lacking is the square of $3a$, that is, the square of one half the coefficient of x . $\left(\frac{6a}{2}\right)^2 = (3a)^2$, or $9a^2$.

Hence $9a^2$ is the required term, since $x^2 + 6ax + 9a^2 = (x + 3a)^2$.

EXERCISES

Add terms to the following expressions to make them squares. Factor the resulting expressions. Do ex. 1 to 10 orally:

- | | |
|-----------------------|-------------------------------------|
| 1. $x^2 + 2x + ()$ | 11. $x^2 + 5x + ()$ |
| 2. $x^2 - 2x + ()$ | HINT. $x^2 + 5x + \frac{25}{4} = ?$ |
| 3. $x^2 + 4x + ()$ | 12. $x^2 - 3x + ()$ |
| 4. $x^2 - 4x + ()$ | 13. $x^2 + 7x + ()$ |
| 5. $x^2 + 6x + ()$ | 14. $x^2 + x + ()$ |
| 6. $x^2 + 8x + ()$ | 15. $x^2 - x + ()$ |
| 7. $x^2 - 8x + ()$ | 16. $x^2 + \frac{5}{3}x + ()$ |
| 8. $x^2 - 10x + ()$ | 17. $x^2 - \frac{3}{2}x + ()$ |
| 9. $x^2 + 12x + ()$ | 18. $x^2 + \frac{3}{7}x + ()$ |
| 10. $x^2 - 12x + ()$ | 19. $x^2 - \frac{2}{3}x + ()$ |

Solving Quadratic Equations by Completing the Square

EXAMPLE 1. Find the roots of the equation $x^2 + 4x - 12 = 0$.

SOLUTION.

$$x^2 + 4x - 12 = 0 \quad (1)$$

Add 12 to each side:

$$x^2 + 4x = 12 \quad (2)$$

Add to each side of (2) the

$$\text{square of } \frac{1}{2} \text{ the coefficient of } x: \quad x^2 + 4x + 4 = 16 \quad (3)$$

$$\text{Take the square root of each side of (3):} \quad x + 2 = \pm 4 \quad (4)$$

$$\text{Whence} \quad x = -2 \pm 4 \quad (5)$$

$$\text{Solve (5):} \quad x = 2 \text{ or } -6$$

CHECK. $2 - 6 = -4$, the coefficient of x in (1) with sign changed.
 $2(-6) = -12$, the known term of (1). Or, check by substitution.

EXAMPLE 2. Find the roots of the equation $2x^2 - 5x + 2 = 0$.

SOLUTION.

$$2x^2 - 5x + 2 = 0 \quad (1)$$

Subtract 2 from each side:

$$2x^2 - 5x = -2 \quad (2)$$

Divide both sides of (2) by 2:

$$x^2 - \frac{5}{2}x = -1 \quad (3)$$

Add the square of $\frac{1}{2}$ the coefficient of x

$$\text{to each side of (3):} \quad x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 = -1 + \frac{25}{16} \quad (4)$$

$$\text{Take the square root of each side of (4):} \quad x - \frac{5}{4} = \pm \frac{3}{4} \quad (5)$$

$$\text{Solve (5):} \quad x = \frac{5}{4} \pm \frac{3}{4} = \frac{8}{4} \text{ or } \frac{2}{4}$$

$$\therefore x = 2 \text{ or } \frac{1}{2}$$

CHECK. The equation in the form for checking is $x^2 - \frac{5}{2}x + 1 = 0$.
 $2 + \frac{1}{2} = 2\frac{1}{2}$, or $\frac{5}{2}$, and $2(\frac{1}{2}) = 1$. Hence the roots check.

EXAMPLE 3. Find the roots of the equation $x^2 + 4x + 1 = 0$.

SOLUTION.

$$x^2 + 4x + 1 = 0 \quad (1)$$

Subtract 1 from each side of (1):

$$x^2 + 4x = -1 \quad (2)$$

Add the square of $\frac{1}{2}$ the coefficient of x

$$\text{to each side of (2):} \quad x^2 + 4x + 4 = 3 \quad (3)$$

$$\text{Take the square root of each side of (3):} \quad x + 2 = \pm \sqrt{3} \quad (4)$$

$$\text{Solve (4):} \quad x = -2 \pm \sqrt{3}$$

$$\therefore x = \begin{cases} -2 + \sqrt{3} \\ -2 - \sqrt{3} \end{cases}$$

The decimal values of the roots are $x = -2 \pm 1.732^+ = -.268$ or -3.732 .

CHECK. $-2 + \sqrt{3} + (-2 - \sqrt{3}) = -4$ and $(-2 + \sqrt{3})(-2 - \sqrt{3}) = 1$

Also, $-.268 - 3.732 = -4$ and $(-.268)(-3.732) = 1.100116$, or 1.1 (approximately 1).

Summary of steps.

1. Arrange the equation in the form $x^2 + bx = c$, the coefficient of x^2 being 1.
2. Complete the square of the left member of the equation $x^2 + bx = c$ by adding to each side of the equation the square of one half the coefficient of x .
3. Find the square root of each side of the equation. Write the sign $\pm$ before the square root of the right side.
4. Solve the two resulting linear equations.
5. Check the solution.

EXERCISES

Solve the following equations by completing the square. Leave irrational results in the simplest radical form :

- | | |
|-------------------------|---------------------------|
| 1. $x^2 + 2x - 3 = 0$ | 22. $2x^2 - 7x + 2 = 0$ |
| 2. $x^2 + 4x - 5 = 0$ | 23. $4x^2 - 2x = 1$ |
| 3. $x^2 - 6x + 5 = 0$ | 24. $3x^2 + 2x = 1$ |
| 4. $x^2 + 8x + 7 = 0$ | 25. $2x^2 + 5x + 1 = 0$ |
| 5. $x^2 - 8x - 20 = 0$ | 26. $5x^2 + 3x - 2 = 0$ |
| 6. $x^2 - 6x + 3 = 0$ | 27. $3x^2 - 5x + 2 = 0$ |
| 7. $x^2 - 8x = 5$ | 28. $3x^2 + 4x + 1 = 0$ |
| 8. $x^2 + 8 = 6x$ | 29. $4x^2 + 5x + 1 = 0$ |
| 9. $x^2 + 2x = 2$ | 30. $3x^2 + 2x - 1 = 0$ |
| 10. $x^2 + 2x = 5$ | 31. $4x^2 - 3x = 1$ |
| 11. $x^2 + 5x + 4 = 0$ | 32. $2x^2 - 5x = 3$ |
| 12. $x^2 - 5x = 6$ | 33. $5x^2 + 6x + 1 = 0$ |
| 13. $x^2 + 3x - 18 = 0$ | 34. $12x^2 + 13x + 3 = 0$ |
| 14. $x^2 + 3x = 40$ | 35. $x^2 + 2x = 3$ |
| 15. $x^2 + 5x = 7$ | 36. $10x^2 + 5 = 27x$ |
| 16. $x^2 + 3x - 9 = 0$ | 37. $6x^2 - 25x + 14 = 0$ |
| 17. $x^2 + 5 = 7x$ | 38. $8x^2 - 14x = 15$ |
| 18. $x^2 - 2x = 4$ | 39. $x^2 + 3x - 3 = 0$ |
| 19. $x^2 + 4x = 1$ | 40. $x^2 + px + q = 0$ |
| 20. $x^2 - 4x = 3$ | 41. $rx^2 + px + q = 0$ |
| 21. $3x^2 - 6x + 1 = 0$ | 42. $ax^2 + bx + c = 0$ |

HINT. See example 2, page 253.

Solving Quadratic Equations by the Quadratic Formula

The equation $ax^2 + bx + c = 0$ (ex. 42, page 254) is the standard, or general, form of all quadratic equations. Its solution is important and should be remembered.

The result, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, that was found as the roots of the equation $ax^2 + bx + c = 0$ is known as the **quadratic formula** and may be used to find the roots of any quadratic equation.

Any quadratic equation may be put in the standard form $ax^2 + bx + c = 0$. Note that a is the coefficient of x^2 , b is the coefficient of x , and c is the known term.

EXAMPLE 1. Solve the equation $x^2 - 5x + 4 = 0$.

SOLUTION. In this equation a , the coefficient of x^2 , is 1, b , the coefficient of x , is -5 , and c , the known term, is 4. To solve the equation substitute the values of a , b , and c in the formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{5 \pm \sqrt{25 - 16}}{2} = \frac{5 \pm 3}{2} = 4 \text{ or } 1 \end{aligned}$$

CHECK. Check by substitution or by finding the relations between the roots as given on page 250.

EXAMPLE 2. Solve the equation $3x^2 + 4x = 5$.

SOLUTION. $3x^2 + 4x = 5$

Subtract 5 from both sides: $3x^2 + 4x - 5 = 0$

Substitute $a = 3$, $b = 4$, and $c = -5$ in the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(3)(-5)}}{2 \times 3} = \frac{-4 \pm \sqrt{76}}{6} \\ &= \frac{-4 \pm 2\sqrt{19}}{6} = \frac{-2 \pm \sqrt{19}}{3} \\ &= \frac{-2 + \sqrt{19}}{3} \text{ or } \frac{-2 - \sqrt{19}}{3} \end{aligned}$$

CHECK. Check by substitution in the original equation or by finding the sum of the roots and their product.

EXERCISES

Solve by the formula. Leave irrational results in radical form :

1. $4x^2 - 13x + 3 = 0$

9. $x^2 = x + 2$

2. $10x^2 + 10 = 29x$

10. $7x^2 = 4x + 3$

3. $10x^2 + 99x = 10$

11. $24x^2 + 2x - 5 = 0$

4. $15x^2 - 34x + 15 = 0$

12. $3x^2 - 7x - 8 = 0$

5. $2x^2 - 15x + 25 = 0$

13. $2x^2 - 8x + 3 = 0$

6. $3x^2 - 14 = x$

14. $3x^2 - 11x + 10 = 0$

7. $5x^2 + 13x = 6$

15. $x^2 - x = 210$

8. $x^2 + x = 1$

16. $4.2x^2 - 3.9x = 3$

Find the result correct to the nearest tenth :

17. $x^2 + 2x = 13$

21. $x^2 + 12x = 40$

18. $x^2 + 3x = 5$

22. $x^2 - 2x = 2$

19. $3x^2 + 6x = 15$

23. $x^2 + 1.2x = 23$

20. $2x^2 - 20x = 12$

24. $x^2 - 2.5x = 3$

The three methods of solving quadratic equations.

1. Solve by the *factoring method* whenever the expression is easily factored.

Thus,

$$x^2 - 3x - 4 = 0$$

$$(x - 4)(x + 1) = 0$$

$$x = 4 \text{ or } -1$$

2. Solve by the method of *completing the square* when the coefficient of x^2 is 1 and the coefficient of x is an even number.

Thus,

$$x^2 - 8x = 5$$

$$x^2 - 8x + 16 = 21$$

$$x - 4 = \pm \sqrt{21}$$

$$x = 4 \pm \sqrt{21}$$

As this is the general method for the solution of all quadratic equations in one unknown number and is also the basis for the quadratic formula, you should be thoroughly familiar with it.

3. Solve by the *quadratic formula* whenever methods 1 and 2 are not easily applicable.

EXERCISES

Solve the following equations. Give irrational results correct to the nearest hundredth:

1. $n^2 - 4n = 0$
2. $n^2 - 20n + 91 = 0$
3. $6n^2 - 17n = 133$
4. $3x^2 - 13x + 4 = 0$
5. $x^2 - x = \frac{3}{8}$
6. $x^2 + 8x - 14 = 0$
7. $3x^2 - 7x - 20 = 0$
8. $3x^2 - 5x - 2 = 0$
9. $x^2 = 5x$
10. $6n^2 + 5n = 56$
11. $\frac{n}{n-1} + \frac{n-1}{n} = 2\frac{1}{2}$
12. $\frac{n}{n+1} + \frac{n+1}{n} = 2\frac{1}{6}$
13. $\frac{8}{x-4} = \frac{15}{x-2} + 3$
14. $\frac{x^2 - 5x}{x+3} = x - 3 + \frac{1}{x}$
15. $\frac{3x}{5} = \frac{x^2}{2} - \frac{28}{5}$
16. $\frac{x+2}{x-7} - \frac{x+5}{x-5} = 1$
17. $\frac{4x+5}{7x-1} = \frac{x+2}{5x-3}$
18. $\frac{5-4x}{2x+1} + 3 = \frac{3x-1}{7-x}$
19. $\frac{6}{x-6} - \frac{6}{x-3} = \frac{5}{x-4}$
20. $\frac{6}{n} - \frac{2}{n-1} = \frac{1}{n-2}$
21. $2x^2 - 24x = 56$
22. $x^2 - 24 = -10x$
23. $(2x+1)(x-2) = 2$
24. $(2n-1)(n-3) = 2$
25. $3n^2 - 4.16 = 1.2n$
26. $t^2 - 1.8t = 1$
27. $35t^2 - 3t = 2$
28. $2t^2 - 12t = 54$
29. $3r^2 + 27r = 66$
30. $x^2 + x + \frac{2}{9} = 0$
31. $\frac{4}{3x-5} = 3 + \frac{5}{2x+1}$
32. $\frac{10-r}{3} = \frac{8}{r}$
33. $\frac{x+2}{x+4} = \frac{5x+1}{x+9}$
34. $\frac{2x-6}{x+1} + \frac{24}{x^2-1} = \frac{x+2}{x-1}$
35. $\frac{x-2}{x+2} + \frac{3}{2} = \frac{x+2}{x-2}$
36. $\frac{5}{3m-5} + \frac{m}{m-15} = 0$
37. $\frac{8}{m+1} + 5 = \frac{12}{m-2}$
38. $\frac{3x-1}{7-x} - \frac{5-4x}{2x+1} = 3$
39. $\frac{2y-5}{3y+5} = \frac{y+1}{5y-1}$
40. $\frac{x+2}{x-1} = \frac{7}{3} - \frac{x-4}{2x}$

Exercises with Literal Equations and Formulas

Solve for x :

1. $x^2 + 3a^2 = 4ax$

5. $(1 - a^2)x^2 = b(2x - b)$

2. $x^2 = (m + n)x$

6. $(x + a + m)^2 = (a - m)^2$

3. $x^2 + 2ax = b$

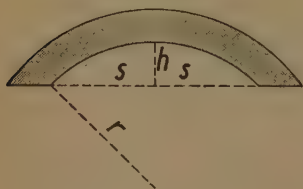
7. $(a - b)x^2 = a^2(x - b)$

4. $4x^2 - 4ab = b^2 - 8ax$

8. $ax^2 = b$

9. (a) Solve the formula $S = a + vt - 16t^2$ for t .(b) Find t when $S = 46$, $a = 32$, and $v = 60$.10. (a) Solve the formula $S = \frac{n}{2}[2a + (n - 1)d]$ for n .(b) Find n when $S = 820$, $a = 3$, and $d = 4$.

11. If the radius of a circle is r , the side s of a regular decagon (10 sides) inscribed in the circle is determined from the proportion $\frac{r}{s} = \frac{s}{r - s}$. Solve for s . Determine the value of s to the nearest tenth when $r = 10$.



12. The radius of a circular arch is given by the formula $r = \frac{s^2 + h^2}{2h}$.

Find the height, h , of the arch if $r = 30$ and $s = 20$. Are both results possible? Explain.

13. (a) Solve the formula $P = \frac{nd^2}{2.5}$ (ex. 52, page 10) for d .

(b) Find the value of d when $P = 78$ and $n = 6$.

14. (a) Solve the formula $m = \frac{gv^2}{c}$ for v .

(b) Find v correct to tenths if $m = 12$, $c = 3.3$, and $g = 15$.

15. The length of a pendulum is proportional to the square of the number of seconds required for a complete oscillation.

$\frac{L}{l} = \frac{T^2}{t^2}$. Solve for t . Find t if $T = 1$, $L = 39.1$, and $l = 19.55$.

Problems

1. Find a number whose square is 56 greater than itself.

HINT. $n^2 = n + 56$. Explain. Show that there are two such numbers.

2. Find a number such that its square is 72 less than 17 times the number.

3. Find a number such that the product of the number and the number diminished by 2 is 15.

4. Find a number such that the product of the number and the number increased by 10 is 75.

5. If 6 times the square of a certain number is equal to the number increased by 35, what is the number?

6. Three times the square of a certain number is equal to 50 decreased by 5 times the number. Find the number.

7. The sum of two numbers is 25 and their product is 144. What are the numbers?

8. The difference between two numbers is 12 and their product is 108. What are the numbers?

9. The difference between two numbers is 8 and the sum of their squares is 274. Find the numbers.

10. Find two consecutive integers whose product is 462.

HINT. There are two sets of numbers.

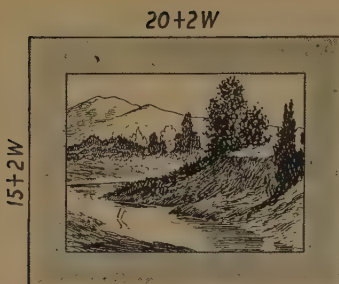
11. Find two consecutive integers the sum of whose squares is 85.

12. Find the price of eggs per dozen when 1 more egg for 6¢ reduces the price of a dozen eggs 12¢.

13. Find the price of eggs per dozen when 3 less for 36¢ raises the price 12¢ a dozen.

14. The side of one square sheet of paper is 2 in. longer than the side of another square sheet. Find the side of each square if the sum of their areas is 340 sq. in.

15. The perimeter of a rectangular building lot is 260 ft. and its area is 2625 sq. ft. Find the dimensions of the lot.



16. A picture 20 in. by 15 in. is surrounded by a frame. Find the width of the frame if its area is 246 sq. in.

17. How wide a strip must be cut around a rectangular lawn 80 ft. by 60 ft. in order to mow $\frac{1}{2}$ of it?

18. A garden plot 50 ft. by 30 ft. is surrounded by a path of uniform width. The area of the path is 516 sq. ft. What is the width of the path?

19. A circular park with a radius of 50 ft. is surrounded by a path of uniform width. If the area of the path is $\frac{1}{2}\frac{1}{5}$ of the area of the park, find the width of the path. (Area of circle = πr^2 .)

20. A path 5 ft. wide surrounds a circular park. If the area of the path is $\frac{1}{3}\frac{3}{8}$ of the area of the park, find the diameter of the park.

21. The diagonal of a rectangular lot is 130 ft. Find the dimensions of the lot if one side is 70 ft. longer than the other side.

22. A rectangle is 12 in. wide by 16 in. long. How much must be added to the length to increase the diagonal by 4 in.?

23. An open box is made from a piece of tin that is 5 in. longer than it is wide by cutting out a 5-inch square from each corner and turning up the sides. If the box contains 750 cu. in., what are the dimensions of the original piece of tin?



24. Mr. Stanley received \$864 for radio sets. If the price of each set had been \$3 more, he would have sold 4 sets less. Find the price of the radio sets and the number of sets he sold.

25. Victor earned \$288 during the summer. If he had earned \$2 a day less, he would have had to work 12 days more to earn the same amount. How many days did he work? How much did he earn a day?

26. A group of men agreed to pay \$147 for a dinner; but as 7 of the men failed to attend, each of those who did attend had to pay \$.50 more. How many men attended the dinner?

27. Mr. Schuyler bought a number of row boats for \$1500. He sold all but 2 of them at an advance of \$25 on each boat and received the same amount that he paid for all of them. How many boats did he buy?

28. Mr. Schuyler bought a secondhand row boat. After repairing and painting it, he sold it for \$75. The per cent of his gain on the cost was equal to the number of dollars he paid for the boat. How much did the boat cost him?

29. Miss Jones bought some household goods at an auction and immediately sold them for \$48, with a per cent of gain on the cost equal to $\frac{1}{2}$ the number of dollars she paid for the goods. How much did she pay for the goods?

30. In a village telephone system each subscriber may talk with each other subscriber; that is, if there are n subscribers, the total number of different calls that may be made is $n(n - 1)$. Find the number of subscribers if 2352 calls may be made.

31. A crew can row 4 mi. down a stream and back again in 1 hr. 50 min. If the rate of the stream is $2\frac{1}{2}$ mi. an hour, what is the rate of the crew in still water?

32. Two men together can do a piece of work in 6 days. How long will it take each to do the work alone if it takes one 5 days longer than it does the other?

33. A certain cistern can be filled by two pipes, running together, in $3\frac{3}{4}$ hr.; but the larger pipe alone will fill it in 4 hr. less time than the smaller one. What time is required by each alone to fill it?

Systems of Equations with One Linear Equation and One Quadratic Equation

An equation of the second degree in two variables is a **quadratic equation**.

Thus, $x^2 + y = 5$, $x + y^2 = 3$, and $xy + x = 4$ are quadratic equations.

Every system of simultaneous equations involving one linear equation and one quadratic equation can be solved by the method of substitution. (See page 143.)

EXAMPLE. Solve the system:
$$\begin{cases} 2x - y = 4 \\ xy = 6 \end{cases}$$

SOLUTION.

$$2x - y = 4 \quad (1)$$

$$xy = 6 \quad (2)$$

Solve (1) for y :

$$y = 2x - 4 \quad (3)$$

Substitute $2x - 4$ for y in (2):

$$x(2x - 4) = 6 \quad (4)$$

$$2x^2 - 4x = 6 \quad (5)$$

$$x^2 - 2x - 3 = 0 \quad (6)$$

$$(x - 3)(x + 1) = 0 \quad (7)$$

Substitute the values of x in (3):

$$\left. \begin{aligned} x &= 3 \text{ or } -1 \\ y &= 2 \text{ or } -6 \end{aligned} \right\}$$

CHECK. Substitute each pair of values of x and y in both of the original equations.

$$\left. \begin{aligned} 2(3) - (2) &= 4 \\ (3)(2) &= 6 \end{aligned} \right\}$$

$$\left. \begin{aligned} 2(-1) - (-6) &= 4 \\ (-1)(-6) &= 6 \end{aligned} \right\}$$

Note that (2) could be solved for either x or y and the resulting value of the variable could be substituted for that variable in (1).

When stating the roots of a system of quadratic equations, be careful to *group the pairs of values correctly*.

Summary of steps.

1. Solve one of the equations for one of the variables in terms of the other variable. Substitute this value of the variable found in the other equation. Solve the resulting equation.

2. Substitute the roots found in step 1 for the variable in one of the original equations. Solve the resulting equation for the remaining variable. Group the pairs of roots found.

3. Check by substituting each pair of values found for the variables in both of the original equations.

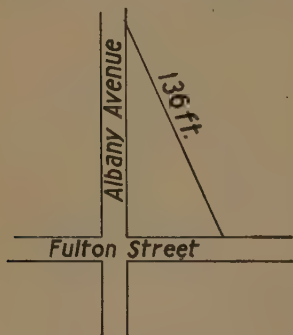
EXERCISES

Solve the following equations. Check ex. 1 to 10:

1. $\begin{cases} x + y = 5 \\ xy = 6 \end{cases}$
2. $\begin{cases} x + y = 5 \\ xy = -6 \end{cases}$
3. $\begin{cases} xy = 8 \\ x - y = 2 \end{cases}$
4. $\begin{cases} x - 8 = 0 \\ x^2 + y^2 = 68 \end{cases}$
5. $\begin{cases} x + y = 8 \\ 3x^2 - 5y = 60 \end{cases}$
6. $\begin{cases} x^2 - y^2 = 32 \\ 3x + 5y = 8 \end{cases}$
7. $\begin{cases} x + y = 5 \\ x^2 + xy = 30 \end{cases}$
8. $\begin{cases} 2x + 3y = 2 \\ 4x^2 = 9y^2 \end{cases}$
9. $\begin{cases} 2x + y = 11 \\ xy = 2x - 35 \end{cases}$
10. $\begin{cases} 2x^2 + xy = -1 \\ x = y - 4 \end{cases}$
11. $\begin{cases} 5x - 3y = 7 \\ x^2 + xy + y^2 = 7 \end{cases}$
12. $\begin{cases} 3x + y = 2 \\ xy = x - 24 \end{cases}$
13. $\begin{cases} 2x + y = 2 \\ y^2 = 11x + 9 \end{cases}$
14. $\begin{cases} x^2 + y^2 = 25 \\ y = x + 1 \end{cases}$
15. $\begin{cases} 3x - 7y = 6 \\ xy = 15 \end{cases}$
16. $\begin{cases} 2x - 8y = 2 \\ xy + y^2 = 18 \end{cases}$
17. $\begin{cases} 2x^2 - y^2 = -2 \\ 3y = x + 7 \end{cases}$
18. $\begin{cases} 7x + 2y = -8 \\ x - xy = 4 \end{cases}$
19. $\begin{cases} x + 3y = 6 \\ x^2 - 5y^2 = 4 \end{cases}$
20. $\begin{cases} x + 5y = 17 \\ 2x^2 + y^2 = 17 \end{cases}$
21. $\begin{cases} x^2 - y^2 = 40 \\ 2x + y = 17 \end{cases}$
22. $\begin{cases} x^2 + y^2 = 17 \\ x = 3y + 1 \end{cases}$
23. $\begin{cases} \frac{x+y}{y} = 6 \\ xy = 45 \end{cases}$
24. $\begin{cases} \frac{x}{y} + \frac{y}{x} = 2 \\ 6x - 5y = 1 \end{cases}$
25. $\begin{cases} \frac{1}{y} - \frac{1}{2} = \frac{1}{x} \\ x - 3y = -\frac{5}{8} \end{cases}$
26. $\begin{cases} \frac{x+1}{y+1} = \frac{3}{4} \\ \frac{x^2+y}{x+y^2} = \frac{7}{11} \end{cases}$
27. $\begin{cases} (x-4)(y-7) = 0 \\ x + y = 12 \end{cases}$

Problems

1. Find two numbers whose sum is 20 and whose product is 96.
2. Find two numbers whose sum is 18 and the sum of whose squares is 234.
3. The difference between two numbers is 8 and the difference between their squares is 96. Find the numbers.
4. A certain fraction increased by its reciprocal is $\frac{13}{8}$. The sum of the numerator and the denominator of the fraction is 5. Find the fraction.
5. The ratio of two numbers is 3 to 4 and $\frac{1}{3}$ of the product of the numbers is 64. Find the numbers.
6. Find two numbers in the ratio of 5 to 3 such that the difference of their squares shall be 36.
7. The area of a rectangular field is 4 A. Find its dimensions if it takes 104 rd. of fencing to inclose it.
8. The perimeter of an athletic field is 700 ft. and the diagonal is 250 ft. Find the dimensions of the field.



9. A rectangular playground is 55 ft. longer than it is wide and its area is 60,500 sq. ft. Find its dimensions.
10. A city lot situated at the intersection of two perpendicular streets is triangular in shape. It has a total street frontage of 184 ft. and a back line of 136 ft. Find the other two sides of the lot. (See page 9.)
11. The yearly interest on a certain sum of money is \$42. If the sum were \$200 more and the rate 1% less, the annual income would be \$6 more. Find the principal and the rate.

HINT. $\frac{Pr}{100} = 42$; $(P + 200)\frac{(r - 1)}{100} = 48$. Eliminate the Pr term.



Euclid

CHAPTER XV

FIRST STEPS IN DEMONSTRATIVE GEOMETRY



Pythagoras

Historical Introduction

In your study of arithmetic and algebra, you have had to work problems concerning triangles, rectangles, circles, and other figures. Algebra is concerned chiefly with the study of the equation and with the solution of problems by means of the equation. Geometry, on the other hand, deals with lines, their combinations to form different figures, and the relations between these figures.

The study of geometry is supposed to have originated with the Egyptians. Each year the Nile river rose and overflowed its banks and flooded the surrounding country. This made the Nile valley very fertile, but it also caused considerable damage; for it washed away sections of the fields and covered up important landmarks. The necessity for finding methods of measuring the land led the Egyptians to the study of geometry, which literally means "earth measurement." With such knowledge as a basis, the Egyptians built palaces and pyramids which are still standing. The practical applications interested them so much that they extended the study of geometry until they had discovered a great many other facts.

The Greeks followed the Egyptians in this study and three hundred years before Christ, a Greek named Euclid wrote a textbook that contains much of the material that is still used in our textbooks to-day. He did so much for the study of geometry that in England this study is called *Euclid* instead of *geometry*.

Review of Terms and Constructions

Let us begin our study of geometry with the line. We shall consider two kinds of lines — straight lines and curved lines.

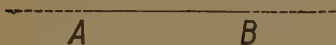
A **straight line** is the shortest line that can be drawn between two points.

The edge of your ruler or the edge of this page in your book represents a straight line (st. line).

A **curved line** is one no part of which is straight.

Point out examples of curved lines in the classroom.

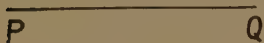
In general, a straight line is called simply a **line**. Unless otherwise limited, a line is indefinite in extent. That is, it extends in both directions without limit. Thus, AB is a straight line.



The lines and figures that are studied in elementary geometry all lie on a flat, smooth surface called a **plane**.

Thus, the surface of the blackboard or of your desk is a plane surface.

EXERCISE 1. Locate two points P and Q . With the use of your ruler or some other straight edge, draw a straight line between the two points. Can you draw more than one straight line between the two points?



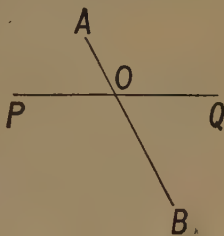
EXERCISE 2. If an aviator wished to fly from one city to another city in the quickest possible time, what kind of course would he choose? Why? How many such courses would there be?

EXERCISE 3. Fasten a thumb tack in the corner of your desk and another tack in the desk across the aisle. Stretch a thread from one tack to the other tack. Do the same thing with another thread. What relation do the two threads bear to each other?

These exercises show us the fact that: *Through two points only one straight line can be drawn.*

Since this is true, it is sometimes stated that: *Two points determine a straight line.*

EXERCISE 4. Draw any straight line, as the line AB . Draw any other straight line, as PQ , intersecting AB at some point O . How many times will the two lines cross if each is extended indefinitely in both directions? How many such lines can you draw through O ?



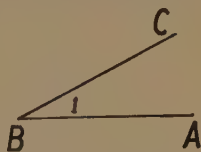
EXERCISE 5. If you know that Mr. Brown lives at the intersection of Albany Avenue and Crown Street, can you find his house easily? Why?

Two straight lines can intersect in only one point.

That is:

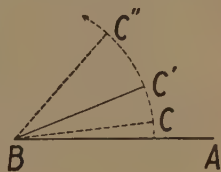
Two intersecting lines determine a point.

Angles. If two straight lines are drawn from the same point, as lines BA and BC , an **angle** is formed. This angle is read "angle ABC " or "angle CBA ." B is the **vertex** of the angle and is read as the middle letter.

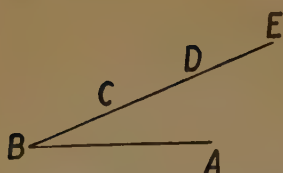


The abbreviation for *angle* is the sign $\angle$. Sometimes angles are designated by one letter or numeral, as $\angle 1$ above.

The line BA may be called the initial line, while the line BC may be considered as rotating counterclockwise starting from the position BA and assuming any one of many positions, as BC , BC' , BC'' , etc. The amount of rotation from the initial position determines the size of the angle.



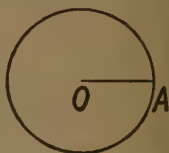
Thus, $\angle ABC$ is less than $\angle ABC'$ or $\angle ABC''$.



The lengths of the sides of an angle have nothing to do with the size of the angle.

Thus, $\angle ABC = \angle ABD = \angle ABE$.

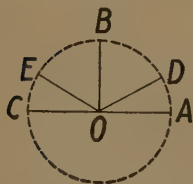
Circles. If the line OA makes a complete rotation about the point O as a center, the point A describes a **circle**. The curved line is called the **circumference** of the circle. It is the location or **locus** (see page 133) of all points that are situated at a given distance from the point O .



The line OA is a **radius** (plural, *radii*) of the circle.

The circumference of a circle is divided into 360 degrees (360°). This is based on the ancient Babylonian division which was made on the supposition that there were 360 days in the year. The degree is divided into 60 smaller units called *minutes* ($'$) and each minute is divided into 60 parts called *seconds* ($''$).

Measurement of angles. If the rotating side of an angle has made one quarter of a complete revolution, the angle formed is called a **right angle**.



Thus, $\angle AOB$ is a right angle (rt. $\angle$).

If the rotating side of an angle has completed one half of a revolution, the angle formed is called a **straight angle**.

Thus, $\angle AOC$ is a straight angle (st. $\angle$). The sides AO and OC form one straight line, AOC .

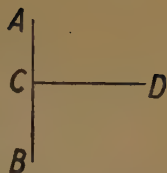
An angle less than a right angle is called an **acute angle**.

An angle greater than a right angle but less than a straight angle is called an **obtuse angle**.

Thus, $\angle AOD$ is an acute angle and $\angle AOE$ is an obtuse angle.

Point out two acute angles in the classroom; four right angles; two obtuse angles; one straight angle.

If one line makes a right angle with another line, the two lines are said to be **perpendicular** to each other.



Thus, if $\angle DCB$ is a rt. $\angle$, then DC is perpendicular to AB ($DC \perp AB$). Also, AB is perpendicular to DC ($AB \perp DC$).

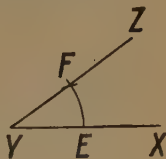
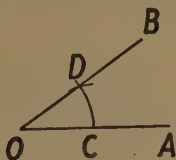
Two straight lines or two angles are *equal* to each other if they have the same size. That is, if one figure can be placed upon the other so that the two figures exactly fit, they are equal and are said to **coincide**.

Thus, line AB is equal to line CD if you can place line AB on line CD so that the point A falls on point C and point B falls on point D .

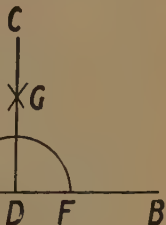
EXERCISES

1. The number of degrees in a right angle is — because —.
2. The angle formed by two perpendicular lines is — degrees.
3. The number of degrees in an acute angle lies between — and —.
4. The number of degrees in a straight angle is —.
5. An angle that lies between 90° and 180° is called —.
6. Explain how $\angle XYZ$

was constructed with compasses and ruler equal to $\angle AOB$.

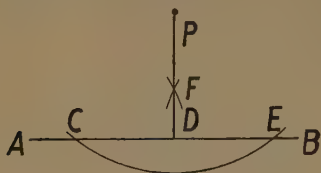


7. Draw an acute angle with sides 2 in. long. Construct another angle equal to it. Cut out the two angles and see if you can make them exactly fit over each other.



8. Explain how CD was drawn perpendicular to AB at the point D on the line AB .

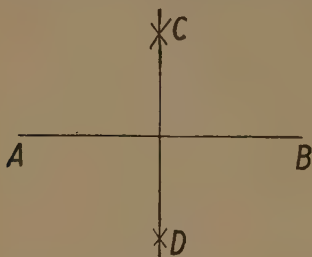
9. Draw a straight line AB . At some point D on AB construct DC perpendicular to AB . Cut out the two angles you have drawn, $\angle ADC$ and $\angle BDC$, and see if they will exactly fit over each other.



10. Explain how the perpendicular PD at the left was drawn from the point P to the line AB .

11. Draw a line 3 in. long. Locate a point outside the line and draw a perpendicular to the line from that point. Draw several other lines from the point to the first line.

12. Measure each of the lines you drew from the point to the given line in ex. 11. Which of these lines is the shortest? Can you draw a shorter line from the point to the given line?

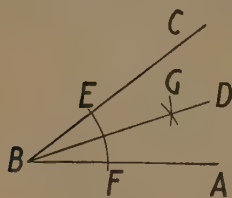


From this experiment we conclude that:

The perpendicular is the shortest line that can be drawn from a point to a straight line.

13. Explain how the perpendicular bisector of line AB was drawn.

14. Draw a line $3\frac{1}{2}$ in. long and construct its perpendicular bisector. Draw the perpendicular bisectors of each half of the line.



15. Explain how the line BD was drawn so as to bisect $\angle ABC$.

16. Draw an acute angle and construct its bisector.

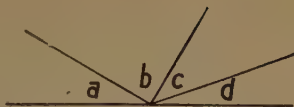
17. Draw an obtuse angle and construct its bisector.

18. What angle is formed when a straight angle is bisected?
A right angle is one half of a straight angle.

From ex. 9 and ex. 18, we conclude that :

All right angles are equal.

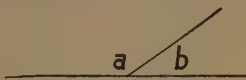
19. What is the sum of $\angle a + \angle b + \angle c + \angle d$?



The sum of all the angles about a point on one side of a straight line is 180° .

New relations between angles. If one line meets another line, as in the figure at the right, two angles are formed whose sum is a straight angle. These angles are called **supplementary angles**.

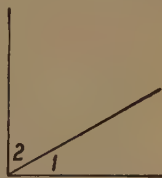
Thus, $\angle a$ and $\angle b$ are supplementary angles.



When the two angles are adjacent, as in the figure above, they are called **supplementary adjacent angles**.

Two angles whose sum is a right angle are called **complementary angles**.

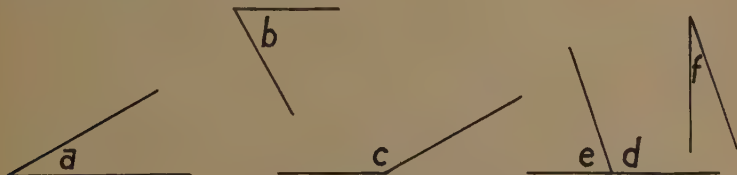
Thus, $\angle 1$ and $\angle 2$ are complementary angles.



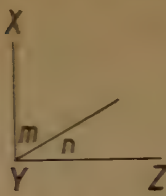
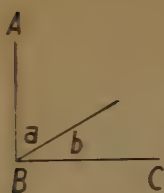
Supplementary angles and complementary angles may have any position. They are not necessarily adjacent to each other, as in the figures above.

EXERCISES

1. Select the pairs of angles from those below that appear to be supplementary angles.



2. Select the pairs of angles from those above that appear to be complementary angles.



3. In the figures at the left, suppose that $\angle ABC = 90^\circ$, $\angle XYZ = 90^\circ$, and $\angle b = \angle n$. What relation exists between $\angle a$ and $\angle m$? between $\angle m$ and $\angle n$?

$$\angle a + \angle b = 90^\circ \text{ and } \angle m + \angle n = 90^\circ$$

Why?

Quantities that are equal to the same quantity or to equal quantities are equal to each other.

Therefore ($\therefore$) $\angle a + \angle b = \angle m + \angle n$

But $\angle b = \angle n$

If equals are subtracted from equals, the results are equal.

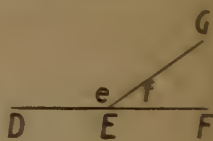
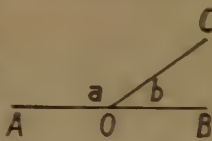
$$\therefore (\angle a + \angle b) - \angle b = (\angle m + \angle n) - \angle n$$

or

$$\angle a = \angle m$$

Complements of the same angle or of equal angles are equal.

4. In the figure at the right, $\angle AOB$ and $\angle DEF$ are straight angles and $\angle b = \angle f$. What relation exists between $\angle a$ and $\angle b$? between $\angle e$ and $\angle f$?



$$\angle a + \angle b = 180^\circ \text{ and } \angle e + \angle f = 180^\circ$$

Why?

$$\therefore \angle a + \angle b = \angle e + \angle f$$

Why?

$$\angle b = \angle f$$

Why?

$$\therefore \angle a = \angle e$$

Why?

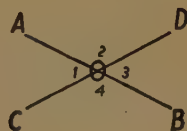


Supplements of the same angle or of equal angles are equal.

Vertical angles. If two straight lines intersect, as in the figure, four angles are formed. Any two opposite angles formed by the prolongation of the lines through the point of intersection are **vertical angles**.

Thus, $\angle 1$ and $\angle 3$ are vertical angles; also $\angle 2$ and $\angle 4$.

You might assume that two vertical angles are equal because they look equal; but you know how deceptive appearances are. You might draw a pair of vertical angles and cut them out to see whether you can prove the angles equal by making one angle exactly fit the other angle. This is a good method of proof; but the one below is better.



5. Prove that the vertical angles in the figure at the right are equal; that is, prove that $\angle 1 = \angle 3$.

STATEMENTS

- (1) $\angle 1$ is the supplement of $\angle 2$.
- (2) $\angle 3$ is the supplement of $\angle 2$.
- (3) $\therefore \angle 1 = \angle 3$.

REASONS

- (1) The sum of $\angle 1 + \angle 2$ is a st. $\angle$.
- (2) The sum of $\angle 3 + \angle 2$ is a st. $\angle$.
- (3) Supplements of the same angle ($\angle 2$) are equal.

The statements in the column at the right are the reasons why the statements given in the column at the left are true.

A reason must be given for every step in the proof.

Theorem. The exercise just proved may be stated more fully thus:

If two angles are vertical angles, they are equal.

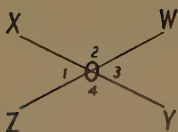
A geometric exercise in this form is called a **theorem** or a **proposition**.

A **theorem** is a statement to be proved. It consists of two parts. First, the facts that are *given*; second, the relation that is *to be proved* true.

The part that is given is sometimes called the **hypothesis** and the part that is to be proved is called the **conclusion**.

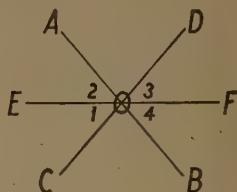
Thus, in the theorem "If two angles are vertical angles, they are equal," it is *given* that the angles are vertical angles. A proof is required that the angles are equal.

EXERCISES



1. In the figure at the left, prove that $\angle 2 = \angle 4$.

2. Prove that if the line EF in the figure at the right bisects $\angle AOC$, it will, if extended, bisect the vertical angle BOD .



Here it is given that $\angle 1 = \angle 2$ and you are to prove that $\angle 3 = \angle 4$.

Show that $\angle 1 = \angle 3$

$\angle 2 = \angle 4$

$\therefore \angle 3 = \angle 4$

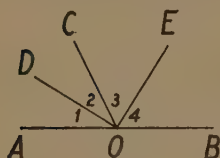
Why?

Why?

Why?

A line that bisects one of two vertical angles will, if extended, bisect the other vertical angle also.

3. The bisectors of two supplementary adjacent angles are perpendicular to each other.



Given: The supplementary adjacent angles AOC and BOC , the line DO bisecting $\angle AOC$, and the line EO bisecting $\angle BOC$.

Prove: $DO \perp EO$.

STATEMENTS

$$\angle 1 = \angle 2$$

$$\angle 4 = \angle 3$$

$$\angle 1 + \angle 4 = \angle 2 + \angle 3$$

$$\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180^\circ$$

$$2(\angle 2 + \angle 3) = 180^\circ$$

$$\therefore \angle 2 + \angle 3 = 90^\circ$$

and

$$DO \perp EO$$

REASONS

Why?

Why?

If equals are added to equals, the sums are equal.

The sum of all the angles about a point on one side of a st. line is 180° .

Substitute $\angle 2 + \angle 3$ for its equal $\angle 1 + \angle 4$.

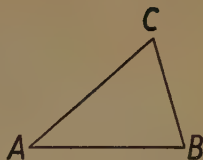
Why?

Why?

Triangles

A **polygon** is a plane figure bounded by straight lines. A polygon formed by three intersecting straight lines is called a **triangle**.

Thus, the figure ABC is a triangle (Δ).

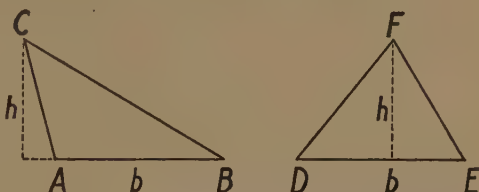


Two triangles that can be made to fit over each other and coincide, part to part, are said to be **congruent** ($\cong$).

Two triangles that have the same area are equal in area, but not necessarily equal in shape. Recall the formula for the area of a triangle, $A = \frac{1}{2}bh$ (see page 5).

The **altitude**, h , of a triangle is the line drawn from the vertex angle perpendicular to the base.

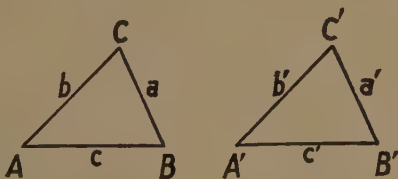
Two triangles having equal bases and equal altitudes have equal areas; but they may have different shapes.



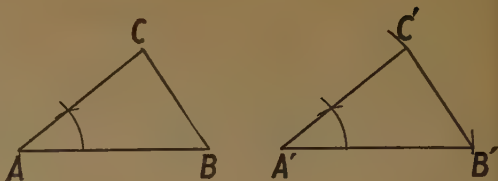
Thus, ΔABC has the same area as ΔDEF , but the two triangles are not congruent.

When two triangles are congruent, it is evident that if one triangle is made to coincide with the other, the three sides and angles of the first triangle will equal the corresponding parts of the second triangle.

Thus, in the figure, $\angle A$ corresponds to and equals $\angle A'$, $\angle B$ corresponds to and equals $\angle B'$, and $\angle C$ corresponds to and equals $\angle C'$. In the same way a corresponds to and is equal to a' , b to b' , and c to c' . Note that side a lies opposite $\angle A$ and similarly for the other sides and angles. Corresponding angles lie opposite corresponding sides.

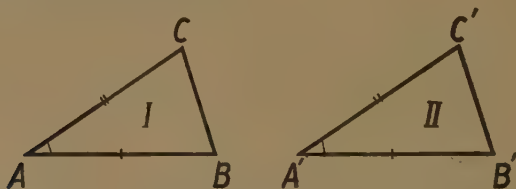


EXERCISE 1. In $\triangle ABC$ and $\triangle A'B'C'$,
 $AC = A'C'$,
 $\angle A = \angle A'$, and
 $AB = A'B'$. Explain how $\triangle A'B'C'$ was constructed.



EXERCISE 2. Construct two triangles having two sides and the included angle of the first equal to the corresponding parts of the other. Cut out the triangles. Can you make them coincide?

Theorem. *Two triangles are congruent if two sides and the included angle of the one are equal to the corresponding sides and the included angle of the other.*



Given: Two $\triangle$, I and II, with $AB = A'B'$, $\angle A = \angle A'$, and $AC = A'C'$.

Prove: $\triangle I$ congruent to $\triangle II$.

Proof: Suppose that you have cut out $\triangle I$ and are going to try to make it coincide with $\triangle II$.

Place $\triangle I$ on $\triangle II$ so that side AB coincides with side $A'B'$ and $\angle A$ coincides with $\angle A'$.

AC then coincides with $A'C'$ and C falls on C' .

Since B falls on B' and C on C' , BC coincides with $B'C'$.

$\therefore \triangle I \cong \triangle II$.

Equal magnitudes can be made to coincide, and AB is given $= A'B'$.

$\angle A$ is given $= \angle A'$.

AC is given $= A'C'$.

Through two points only one st. line can be drawn.

They coincide.

When you clearly *understand* this proof, you can give it with triangles lettered differently from those above. Do not learn the proof by heart, but try to understand the method.

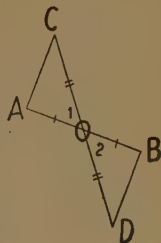
Whenever you find two triangles with two sides and the included angle of one equal to the corresponding parts of the other, the triangles are congruent.

EXERCISES

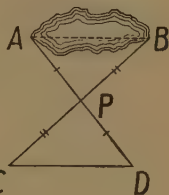
1. In the figure, O bisects lines AB and CD . Prove that $AC = BD$.

HINT. To prove that two lines are equal to each other, show that they are corresponding parts of congruent triangles.

In this case, prove that $\triangle AOC \cong \triangle BOD$. Then show that AC and BD are corresponding parts of the congruent $\triangle$. $\angle 1 = \angle 2$. Why?



2. Explain the measurements that might be taken in the figure at the right to find the distance AB across the pond.

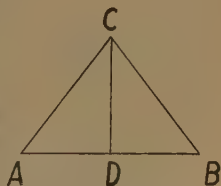


3. If CD is the perpendicular bisector of AB , prove that $CA = CB$.

HINT. Of what two $\triangle$ are CA and CB corresponding parts?

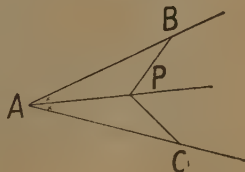
How can you prove two $\triangle$ congruent?

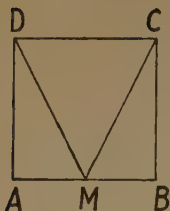
Are there any lines in the two $\triangle$ that you know are equal? Are there any $\angle$ equal? Are these equal $\angle$ included between equal sides? If so, show that the $\triangle$ are congruent.



4. Prove that $\angle CAD = \angle CBD$; that $\angle ACD = \angle BCD$.

5. P is any point in the bisector of $\angle BAC$. $AB = AC$. Prove that P is the same distance from B that it is from C ; also that $\angle BPA = \angle CPA$.





6. Show that the mid-point of the base of a square is equidistant from the opposite vertices.

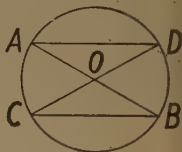
HINT. $AD = BC$. Why? $\angle A = \angle B$. Why?

7. In the figure for ex. 6, prove:

(a) $\angle ADM = \angle BCM$; (b) $\angle MDC = \angle MCD$.

In a circle, a line passing through the center, as AOB , is a **diameter**.

8. If two diameters are drawn as in the figure at the right, prove that lines AD and CB are equal. Name three pairs of equal angles.



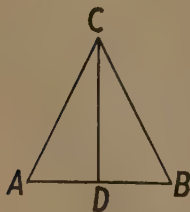
9. If $AB = AC$ and $AD = AE$ in the figure at the right, prove that DC must equal BE .

HINT. DC is part of $\triangle ADC$ and BE is part of $\triangle ABE$. Show that the two $\triangle$ s are congruent. $\angle A$ is common to both of these $\triangle$ s.



10. In ex. 9, show that $\angle B = \angle C$; also that $\angle ADC = \angle AEB$; then that $\angle BDC = \angle CEB$.

An **isosceles triangle** has two sides equal. In $\triangle ABC$, $AC = BC$, $\angle C$ is the vertex angle, $\angle A$ and $\angle B$ are the base angles.



11. Prove that the bisector of the vertex angle of an isosceles triangle divides the triangle into two congruent triangles.

12. Using the fact proved in ex. 11, prove that:

The base angles of an isosceles triangle are equal.

13. Prove that the bisector of the vertex angle of an isosceles triangle is the perpendicular bisector of the base of the triangle.

A **median** of a triangle is a line drawn from the vertex of an angle of the triangle to the mid-point of the opposite side, as CD in the figure for ex. 11.

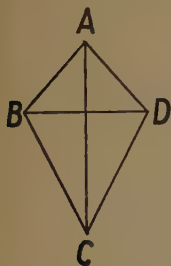
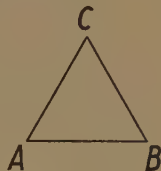
14. How many medians can you draw in a triangle?

15. Prove that if two sides of a triangle are equal, the bisector of their included angle is a median of the triangle.

An **equilateral triangle** has three sides equal.

16. Prove that the three angles of an equilateral triangle are equal to each other.

HINT. Show that $\angle A = \angle B$; then consider AC the base and show that $\angle A = \angle C$.



17. In the kite-shaped figure at the left, prove that $\triangle BCD$ is isosceles if $AB = AD$ and AC bisects $\angle BAD$.

18. In ex. 17, prove that $\angle BCA = \angle DCA$.

19. If, in the figure for ex. 17, $\triangle ABD$ is isosceles and $\triangle CBD$ is isosceles, prove that the line AC divides the figure into two congruent triangles.

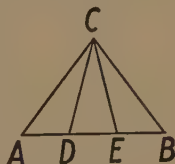
HINT. Show that $\angle ABD = \angle ADB$ and $\angle CBD = \angle CDB$; then $\angle ABC = \angle ADC$.

20. Show that the line AC in ex. 17 is the perpendicular bisector of BD . That is, prove that:

The line that joins the vertices of two isosceles triangles on the same base is the perpendicular bisector of the base.

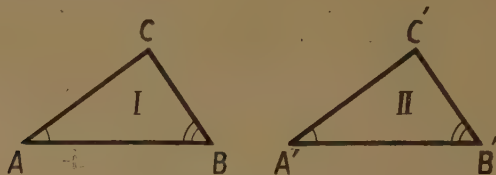
HINT. Use ex. 19 to prove that $\angle BAC = \angle CAD$; then use ex. 13, page 278.

21. If the base AB of isosceles triangle ABC is trisected (divided into three equal parts) at D and E , prove that $CD = CE$.



22. In the figure of ex. 21, show that $\angle CDE = \angle CED$.

Theorem. *Two triangles are congruent if two angles and the included side of one are equal to the corresponding angles and the included side of the other.*



Given: Two $\triangle$, I and II, with $\angle A = \angle A'$, side $AB = A'B'$, and $\angle B = \angle B'$.

Prove: $\triangle I$ congruent to $\triangle II$.

Proof: Suppose that you have cut out $\triangle I$ and are trying to make it coincide with $\triangle II$.

Place $\triangle I$ on $\triangle II$ so that side AB coincides with side $A'B'$ and $\angle A$ coincides with $\angle A'$. AC then falls along $A'C'$ and point C falls somewhere on $A'C'$.

$\angle B$ coincides with $\angle B'$, BC falls along $B'C'$, and C falls somewhere on $A'C'$.

Then point C falls on C' .

$\therefore \triangle I \cong \triangle II$.

Equal magnitudes can be made to coincide and AB is given $= A'B'$.

$\angle A$ is given $= \angle A'$.

$\angle B$ is given $= \angle B'$.

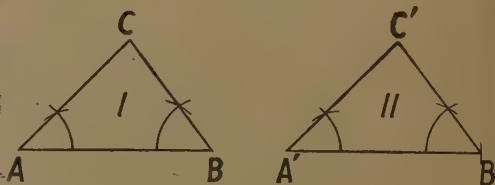
Two st. lines intersect in one point only.

They coincide, point to point.

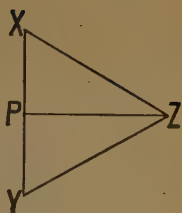
EXERCISES

1. Explain how $\triangle II$ below was constructed if $\angle A' = \angle A$, $A'B' = AB$, and $\angle B' = \angle B$.

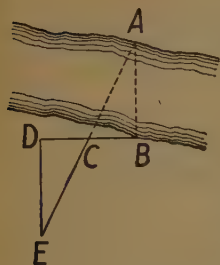
2. Construct two triangles that have bases 3 in. long and equal acute angles at the extremities of the bases.



3. In the figure at the right, PZ is $\perp XY$ and bisects $\angle XZY$. Prove that $\triangle PXZ = \triangle PYZ$.



4. In ex. 3, state each pair of equal lines and equal angles. What kind of triangle is $\triangle XYZ$? Why?

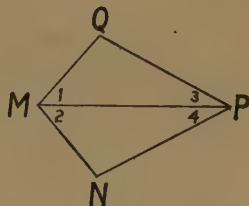


5. Prove that every point in the perpendicular bisector of a straight line is equally distant from the ends of the line.

6. (a) Two boy scouts were directed to find the distance across a river. Explain how they may have used a diagram similar to the figure at the left to find the width of the river.

(b) If $\angle D = 90^\circ$, $DC = 136$ ft., and $EC = 289$ ft., find CB , AC , and AB .

7. If MP bisects $\angle QMN$ and $\angle QPN$, prove that $\angle Q = \angle N$.

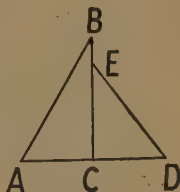


8. (a) State each pair of equal sides in the figure for ex. 7.

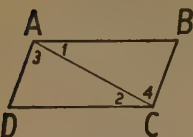
(b) Draw line QN . What kind of triangles are $\triangle MQN$ and $\triangle QNP$?

(c) Prove that $MP \perp QN$. (See ex. 20, page 279.)

9. Show that if BEC is the perpendicular bisector of AD and $\angle A = \angle D$, CE must be the same length as CB and the point E must be the same as point B .



A quadrilateral (quad.) is a polygon having four sides.



10. If in the quadrilateral $ABCD$, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$, prove that $AB = DC$ and $AD = BC$; also that $\angle D = \angle B$.

11. Triangle ABC is isosceles, with $AB = AC$. BE and CD are drawn so that $\angle 1 = \angle 2$. Prove that $BD = CE$.

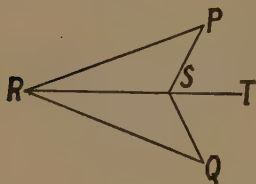
HINT. Show that $\triangle DBC \cong \triangle ECB$.



12. In the figure for ex. 11, prove that:
(a) $\triangle ODB \cong \triangle OEC$; (b) $OB = OC$.

13. In the same figure, prove that if AO is drawn and extended to BC , it will bisect BC . (See ex. 20, page 279.)

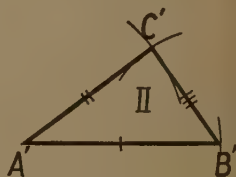
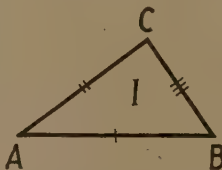
14. In the figure for ex. 11, if $\angle ABE = \angle ACD$, prove that $AE = AD$.



15. In the figure at the left, if RT bisects $\angle PRQ$ and $\angle PSQ$, prove that $RP = RQ$.

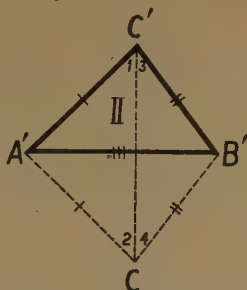
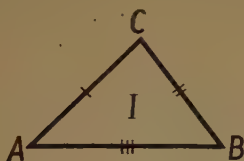
16. In the figure for ex. 15, draw PQ . Prove that RT is the perpendicular bisector of PQ .

17. Explain how $\triangle II$ was constructed if $A'B' = AB$, $A'C' = AC$, and $B'C' = BC$.



18. Construct a triangle similar to $\triangle I$ with $AB = 5$ in., $AC = 4$ in., and $BC = 3\frac{1}{2}$ in. Construct another triangle with sides of the same length. Are these triangles congruent?

Theorem. *Two triangles are congruent if three sides of the one are equal to the corresponding sides of the other.*



Given: Two $\triangle$, I and II, with $AB = A'B'$, $AC = A'C'$, and $BC = B'C'$.

Prove: $\triangle I \cong \triangle II$.

Proof: Suppose that you place $\triangle I$ adjacent to $\triangle II$ so that AB coincides with its equal $A'B'$ and C falls opposite C' . Draw the line $C'C$ to divide the figure into two isosceles $\triangle$.

$\triangle A'C'C$ is isosceles.

Then $\angle 1 = \angle 2$.

$\triangle B'C'C$ is isosceles.

Then $\angle 3 = \angle 4$.

$\angle 1 + \angle 3 = \angle 2 + \angle 4$, or

$\angle 13 = \angle 24$.

$\therefore \triangle II \cong \triangle I$.

$A'C'$ is given $= AC$.

Base angles of isosceles $\triangle$ are $=$.

$B'C'$ is given $= BC$.

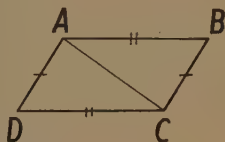
Why?

Equals added to equals give equal sum.

Two sides and included $\angle$ are $=$.

Note that $\angle 1 + \angle 3$ forms $\angle 13$ (read "angle thirteen") as the whole equals the sum of its parts.

EXERCISES



1. If the opposite sides of a quadrilateral are equal, the diagonal divides the quadrilateral into two congruent triangles. State the pairs of equal angles in the figure at the right.

2. In the figure for ex. 15, page 282, if $RP = RQ$ and $PS = SQ$, prove that $\angle PRS = \angle SRQ$.

3. In ex. 2, show that $\angle P = \angle Q$; also that $\angle PST = \angle TSQ$.

Parallel lines. Two lines, as AB and CD , are **parallel** if they lie in the same plane and will not meet however far they are extended. For example, the opposite edges of this page are parallel.

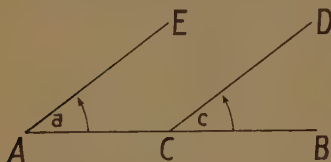
A ————— B

C ————— D

The symbol for "parallel," "parallel to," or "is parallel to" is $\parallel$.

Mention several pairs of parallel lines in the classroom.

Parallel lines may be considered as drawn in the same direction



from the initial line. Thus, the rotation of AE and CD from the initial line AB is the same. That is, $\angle CAE = \angle BCD$.

A line that crosses two parallel lines, as AB crosses AE and CD in the figure above, is called a **transversal**. The $\angle CAE$ and $\angle BCD$ are called **corresponding angles**.

Assume for the present the truth of the two theorems that follow, which you will prove later in your study of geometry.

I. *If two straight lines are cut by a transversal so that the corresponding angles are equal, the lines are parallel.*

II. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Note the difference between these two theorems. Theorem I states that in the figure above, if $\angle a = \angle c$ then $AE \parallel CD$. Theorem II states that if $AE \parallel CD$, then $\angle a = \angle c$. The part that is given in Theorem I becomes the part to be proved in Theorem II, while the part to be proved in Theorem I is the part given in Theorem II.

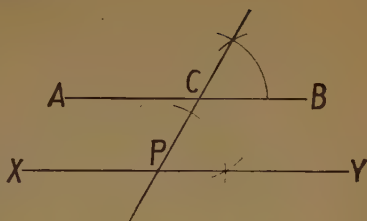
Theorem II is called the **converse** of Theorem I. The converse of a theorem is not always true.

State the converse of:

- (a) If two angles are vertical angles, they are equal.
- (b) The base angles of an isosceles triangle are equal.

EXERCISES

1. Explain how line XY was drawn parallel to line AB through the point P .



2. Draw a line MN and construct a line parallel to MN through a point K outside of MN . How many lines can you draw through K parallel to MN ?

Through a given point, only one line can be drawn parallel to a given line.

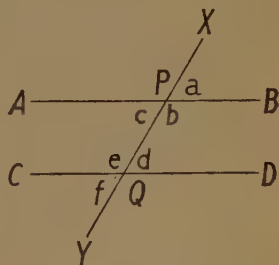
Or, Two intersecting straight lines cannot both be parallel to the same straight line.

3. In the figure at the right, $\angle c$ and d and $\angle b$ and e are called **alternate interior angles** (alt. int. $\angle$). If $\angle c = \angle d$, prove that $AB \parallel CD$.

SOLUTION. $\angle c = \angle a$ Why?

Then $\angle a = \angle d$ Why?

$\therefore AB \parallel CD$ See Theorem I, page 284.



4. In ex. 3, if $\angle b = \angle e$, prove that $AB \parallel CD$.

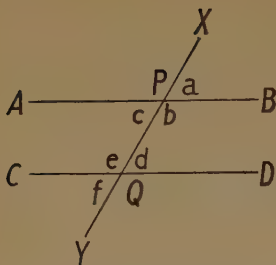
If two straight lines are cut by a transversal so that the alternate interior angles are equal, the lines are parallel.

5. In the figure for ex. 3, prove that AB is parallel to CD if $\angle b + \angle d = 180^\circ$.

HINT. $\angle a$ is the supplement of $\angle b$. Why?

$\angle d$ is the supplement of $\angle b$. Why?

Then $\angle a = \angle d$. Why?



6. If $AB \parallel CD$, prove that $\angle c = \angle d$.

SOLUTION. $\angle c = \angle a$ Why?

$\angle a = \angle d$ See Theorem II,
page 284.

$\therefore \angle c = \angle d$ Why?

Show that this exercise is the converse of ex. 3, page 285.

7. If $AB \parallel CD$ in the figure above, prove that $\angle b = \angle e$.

Of which exercise is this the converse?

If two parallel lines are cut by a transversal, the alternate interior angles are equal.

8. If in the same figure $\angle b$ is supplementary to $\angle d$, prove that $AB \parallel CD$.

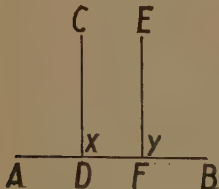
If two straight lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, the lines are parallel.

9. Prove the theorem above if $\angle c + \angle e = 180^\circ$.

10. If $AB \parallel CD$, prove that $\angle b + \angle d = 180^\circ$.

If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.

11. In the same figure, prove that $AB \parallel CD$ if $\angle a = \angle f$.



12. If CD and EF are each $\perp AB$, prove that $CD \parallel EF$.

HINT. $\angle x$ is what kind of angle?

$\angle x = \angle y$ Why?

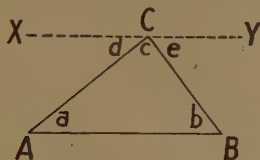
Then $CD \parallel EF$. Why?

Two lines perpendicular to the same line are parallel.

13. If in the figure for ex. 1, page 277, AB and CD bisect each other at O , prove that $AC \parallel DB$.

You have several times made use of the fact that the sum of the angles of a triangle is equal to 180° . You are now going to *prove* this fact. If you have not already cut a triangle into three pieces and fitted the angles together, do so now. (See page 20.)

Theorem. *The sum of the angles of a triangle is 180° .*



Given: $\triangle ABC$, with $\angle A$, B , and C .

Prove: $\angle A + \angle B + \angle C = 180^\circ$.

Proof: To find the sum of the angles, draw the line $XY \parallel AB$ through C and compare the $\angle$ formed at C with $\angle A$ and B .

$$\angle a = \angle d$$

$$\angle b = \angle e$$

$$\text{But } \angle d + \angle c + \angle e = 180^\circ$$

$$\therefore \angle A + \angle C + \angle B = 180^\circ$$

$\angle a$ and d are alt. int. $\angle$ of $\parallel$ lines and alt. int. $\angle$ are =.

Same reason as above.

Sum of the $\angle$ about a point on one side of a st. line = 180° .

A quantity may be substituted for its equal.

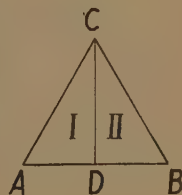
EXERCISES

1. Prove that the acute angles of a right triangle are complementary.

2. What is the value of each angle of an equilateral triangle? (See ex. 16, page 279.)

3. Prove that the altitude CD of the equilateral $\triangle ABC$ divides the triangle into two congruent right triangles.

4. How many degrees are there in $\angle ACD$ and $\angle CAD$ of $\triangle I$, ex. 3?

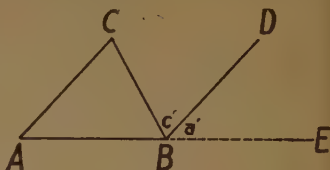


5. Prove that:

The side opposite the 30° angle of a 30° - 60° right triangle is one half the hypotenuse. (See page 240.)

6. Use the figure at the right to prove that the sum of the angles of a triangle equals 180° .

HINT. $BD \parallel AC$.



7. Prove that if two angles of one triangle are equal to two angles of another triangle, then the third angles are equal.

8. Prove that two triangles are congruent if two angles and the side opposite one of these angles of one triangle are equal to the corresponding angles and side of the other triangle.

9. In the figure for ex. 6, if $\angle a' = 28^\circ$ and $\angle c' = 52^\circ$, find the angles of $\triangle ABC$.

10. Find each angle of a triangle if they are in the ratio $2:3:4$.

11. Find each base angle of an isosceles triangle if the vertex angle is 80° .

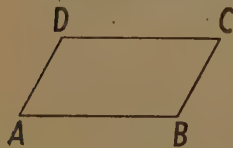
12. Prove that a triangle cannot have more than one right angle or one obtuse angle.

13. Prove that only one perpendicular can be drawn to a straight line from a point without.

14. Find the sum of the angles of a quadrilateral.

Parallelograms

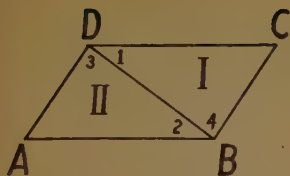
A **parallelogram** is a quadrilateral whose opposite sides are parallel.



Thus, $ABCD$ is a parallelogram ($\square$).

You have worked problems involving parallelograms and have made use of the formula $A = bh$ for the area. (See page 5.)

EXERCISES



1. Prove that the diagonal of a parallelogram divides the figure into two congruent triangles. (See page 5.)

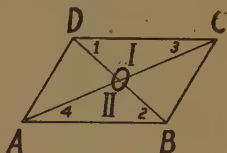
HINT. $\angle 1 = \angle 2$ Ex. 6, page 286.
 $\angle 4 = \angle 3$ Why?
 $\triangle I \cong \triangle II$ Why?

2. Prove that the opposite sides of a parallelogram are equal.

3. Prove that the opposite angles of a parallelogram are equal.

4. Prove that the diagonals of a parallelogram bisect each other.

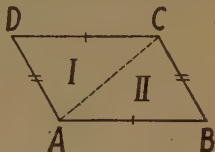
HINT. What is given? Prove that $DO = OB$ and $CO = OA$. Show that they are corresponding parts of congruent triangles.



5. Prove that if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

6. Prove that if the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

HINT. What is given? What is to be proved?
 How may you prove one line $\parallel$ to another line?
 (See ex. 3, page 285.) Show that $\triangle I \cong \triangle II$ and complete the proof.



7. Show that if two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

8. State the converse of ex. 2. Is it true?

9. State the converse of ex. 4. Is it true?

10. Prove that parallel lines included between parallel lines are equal.

11. Find each angle of parallelogram $ABCD$ if $\angle A = 90^\circ$.

Rectangle

A **rectangle** is a parallelogram whose angles are right angles.

A **square** is a rectangle that has all its sides equal.

12. Prove that the diagonals of a square are perpendicular to each other. (See ex. 20, page 279.)

Square

13. If one angle of a parallelogram is a right angle, prove that the figure is a rectangle.



14. Prove that the diagonals of a rectangle are equal.

HINT. Of what two Δ are DB and AC corresponding parts? Can you prove these $\Delta \cong$? Complete the proof.

15. If the diagonals of a parallelogram are equal, the figure is a rectangle.

16. Prove that the bisectors of the opposite angles, A and C , of the rectangle above are equal and parallel.

17. Prove that the median to the hypotenuse of a right triangle equals one half the hypotenuse.

Proof: Rt. ΔABC and the median BD are given. Prove that $BD = \frac{1}{2} AC$.

To show that $BD = \frac{1}{2} AC$, double BD and show that $2BD = AC$.

Ex'tend BD to E , making $DE = BD$. Connect E with A and C .

Quad. $ABCE$ is a parallelogram.

$\square ABCE$ is a rectangle.

Then $AC = BE$, or $2BD$.

That is, $BD = \frac{1}{2} AC$.

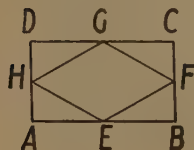


Ex. 5, page 289.

Ex. 13 above.

Ex. 14 above.

Why?



18. If the mid-points of the sides of a rectangle are connected in order, prove that an equilateral parallelogram is formed. That is, show that $HG = GF = FE = EH$.

Similar Triangles

Similar triangles are triangles that have their angles equal and their corresponding sides proportional. They have the same shape, but are not necessarily the same size.

The symbol for *similar* is $\sim$.

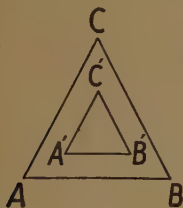


FIG. I

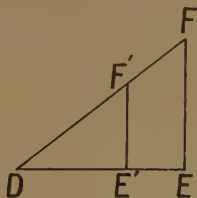


FIG. II

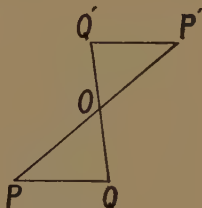


FIG. III

The following pairs of triangles are similar :

$$\triangle ABC \sim \triangle A'B'C', \quad \triangle DEF \sim \triangle D'E'F', \quad \triangle OPQ \sim \triangle OP'Q'$$

In Fig. I, $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$

In Fig. II, $\frac{DE'}{DE} = \frac{E'F'}{EF} = \frac{DF'}{DF}$

In Fig. III, $\frac{P'Q'}{PQ} = \frac{P'O}{PO} = \frac{Q'O}{QO}$

We shall assume that :

Two triangles are similar if the corresponding angles are equal or if the corresponding sides are proportional.

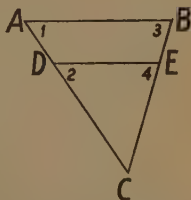
EXERCISES

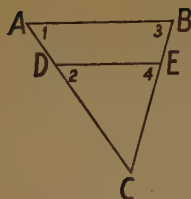
1. If $DE \parallel AB$, prove that $\triangle DEC \sim \triangle ABC$.

HINT. $\angle 1 = \angle 2$. Why? $\angle 3 = \angle 4$. Why? Then the three $\angle$ s of $\triangle ABC =$ the three $\angle$ s of $\triangle DEC$. Why?

2. In the figure for ex. 1, state the corresponding sides of the two similar triangles.

Write the series of equal ratios of the corresponding sides.



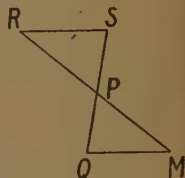


3. If in the figure at the left, DE is given $\parallel AB$, $AB = 8$ in., $AC = 10$ in., $BC = 8$ in., and $DE = 6$ in., find the lengths of DC and EC .

4. In the same figure, if $AB = 24$ in., $AC = 16$ in., $BC = 12$ in., and $DC = 8$ in., find DE and EC .

5. In the same figure, if $AB = 9$ in., $AC = 12$ in., $BC = 10$ in., $DE = 6$ in., $DC = 8$ in., and $EC = 6\frac{2}{3}$ in. are $\triangle ABC$ and DEC similar? Why?

6. If $RS \parallel QM$, prove that $\triangle RSP \sim \triangle MQP$.

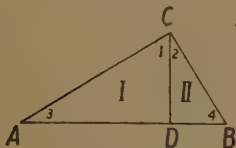


7. State the proportional lines in $\triangle RSP$ and $\triangle MQP$.

8. If in the figure for ex. 6, $RS = 10$ in., $SP = 12$ in., $RP = 15$ in., and $SQ = 20$ in., find QM and PM .

9. In the figure for ex. 6, if $RS = 12$ in., $RP = 15$ in., $PM = 12$ in., and $SQ = 24$ in., find SP , PQ , and QM .

10. Given the right $\triangle ABC$, with $CD \perp AB$. Prove that $\triangle I \sim \triangle ABC$.



HINT. $\angle 1$ and $\angle 3$ are complementary. Why?

$\angle 4$ and $\angle 3$ are complementary. Why?

$\angle 1 = \angle 4$ Why?

Complete the proof.

11. In the figure for ex. 10, prove that: (a) $\triangle II \sim \triangle ABC$; (b) $\triangle I \sim \triangle II$.

12. In the figure for ex. 10, prove that $AD : CD = CD : DB$ or that $\overline{CD}^2 = AD \times DB$. (See ex. 6 page 228.)

NOTE. $\overline{CD}^2$ means "the square of the line CD ."

13. In the figure for ex. 10, page 292, show that:

$$(a) \frac{AD}{AC} = \frac{AC}{AB}, \text{ or } AD \times AB = \overline{AC^2}$$

$$(b) \frac{DB}{CB} = \frac{CB}{AB}, \text{ or } BD \times AB = \overline{CB^2}$$

14. From ex. 13, show that:

$$AD \times AB + BD \times AB = \overline{AC^2} + \overline{CB^2}$$

$$AB(AD + BD) = \overline{AC^2} + \overline{CB^2}$$

$$AB \times AB = \overline{AC^2} + \overline{CB^2}$$

$$\text{or } AB^2 = \overline{AC^2} + \overline{CB^2}$$

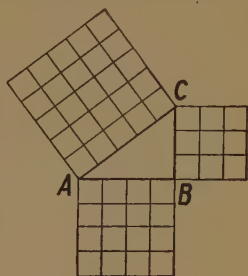
That is,

In any right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides.

The theorem just stated and the one concerning the sum of the angles of a triangle were probably first proved by an influential Greek mathematician, named Pythagoras, who lived more than 500 years before Christ. It is stated that Pythagoras was so elated when he had proved the theorem that bears his name that he sacrificed an ox. Many years before this theorem was proved, its truth was assumed and the fact was used in practical life by the Egyptians and Greeks to construct right angles.

Pythagoras founded very popular schools for the study of mathematics. His most devoted followers formed a brotherhood, which was held together by an oath not to reveal any of the secrets of the organization.

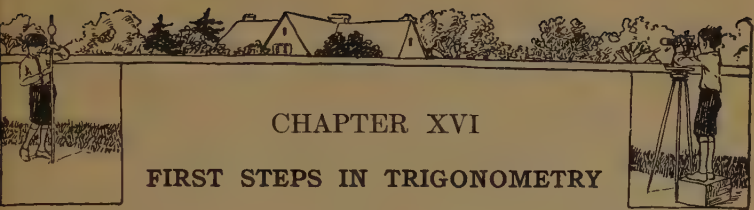
The theorem that bears the name of Pythagoras is one of the most useful facts of geometry. More than 100 proofs have been discovered for this relation. The one given above is an algebraic proof. Show that the relation holds in the figure above, which you have studied in your previous work.



REVIEW EXERCISES

Complete the following statements :

1. Through a given point I can draw — straight lines.
2. Through two given points I can draw — straight line.
3. The size of an angle is determined by —.
4. A right angle is —.
5. Line AB is perpendicular to line CD when —.
6. Two lines are parallel if —.
7. Two geometric magnitudes are congruent if —.
8. The distinction between equal triangles and congruent triangles is —.
9. Two angles are supplementary if —.
10. Two angles are complementary if —.
11. A theorem is —.
12. Every theorem consists of — and —.
13. The converse of a theorem is —.
14. Two triangles are congruent if — or if — or if —.
15. An isosceles triangle is —.
16. The sum of the angles of any triangle equals —.
17. The sum of the acute angles of a right triangle is —.
18. If two parallel lines are cut by a transversal, the — angles are —.
19. Two lines perpendicular to the same line are —.
20. The median of a triangle is —.
21. If one acute angle of a right triangle is 30° , the side opposite is —.
22. A quadrilateral is a parallelogram if two opposite sides are — and — or if the opposite sides are — or —.
23. The diagonals of a rectangle are —.



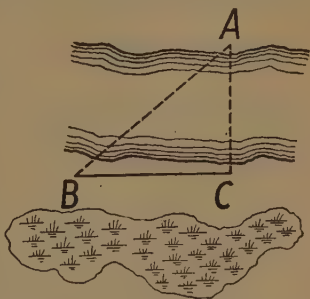
CHAPTER XVI

FIRST STEPS IN TRIGONOMETRY

The foundation of all mathematics lies in measurements. The simplest measurements result from direct counting, as of coins, books, etc. The study of such relations leads from the simplest to the most complicated forms of arithmetic. You have seen how the science of geometry arose from the need to measure the earth's surface. When direct measurements cannot be made, indirect measurements may be taken by means of congruent triangles, as explained in ex. 6, page 281; but if the measurements to be taken are large, this method may be practically as difficult as direct measurement. If the height of a tall object is desired, similar triangles may be used, as explained in ex. 6, page 109; but this method cannot be applied in all cases.

Suppose, for example, that you must find the distance AC across a river from the point C and the only measurements that you can take are the length of BC and the size of the angles B and C . The method you have used in similar triangles will not hold for this case.

Such problems require for their solution new methods, which are explained in **trigonometry**, the science that treats of the measurement of *triangles*.



Similar Triangles

Triangles that have the same shape and equal angles are called **similar triangles** (page 291).

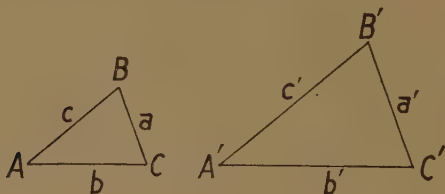
Thus, $\triangle ABC$ and $A'B'C'$ in the figure below are similar. $\triangle ADE$ and ABC in ex. 1 and 6, page 103, also are similar.

Corresponding sides of similar triangles or of equal triangles are sides that are opposite equal angles (see page 275).

In similar triangles, the corresponding sides are proportional.

$$\text{Thus, } \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

$$\text{Also } \frac{a}{c} = \frac{a'}{c'}, \quad \frac{a}{b} = \frac{a'}{b'}, \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'}$$



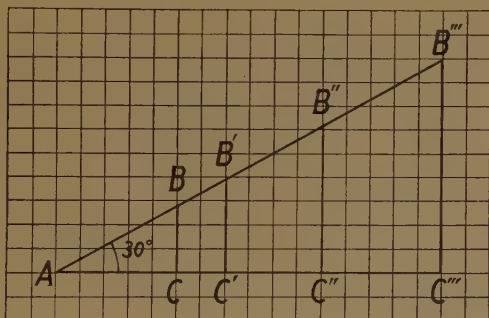
EXERCISES

In the similar triangles ABC and $A'B'C'$, find :

1. Side a if $a' = 6$ in., $b' = 8$ in., and $b = 6$ in.
2. Side b if $a' = 12$ in., $b' = 15$ in., and $a = 20$ in.
3. Side c if $b' = 18$ in., $c' = 24$ in., and $b = 12$ in.
4. Side c' if $a = 5$ ft., $c = 12$ ft., and $a' = 2.5$ ft.
5. The sides of a triangle are 8 ft., 12 ft., and 15 ft., respectively. What are the sides of a similar triangle if the side corresponding to 8 ft. is 10 ft.?
6. If the angles of a triangle are 30° , 40° , and 110° , respectively, what are the angles of a similar triangle twice as large?

Trigonometric Ratios

On squared paper, as in the figure, draw an angle of 30° having the side AC''' 2 in. long. At four places along line AB , as at



B , B' , B'' , and B''' , draw lines perpendicular to AC''' . Carefully measure the lines AB and BC , AB' and $B'C'$, etc. to the nearest $\frac{1}{16}$ of an inch. What is the value of the fractions $\frac{BC}{AB}$, $\frac{B'C'}{AB'}$, etc.? How do these results compare with what you have learned before (page 291)?

In a similar way, construct an angle of 40° , draw the lines perpendicular to one side, and carefully measure the lengths of the side opposite the angle and the hypotenuse to the nearest $\frac{1}{10}$ of an inch. Find the value of the ratio of these lines to the nearest tenth.

If you have done the work accurately, the values of the ratios $\frac{BC}{AB}$, $\frac{B'C'}{AB'}$, etc. should be approximately equal.

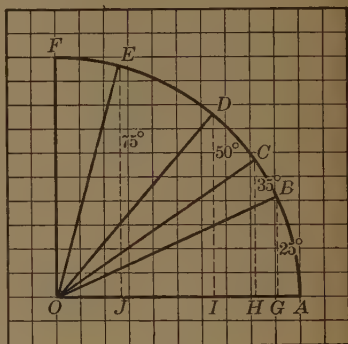
These ratios, and others similar to them, are so important in practical work that special names are given to them. In all right triangles that have the same acute angle, the ratios have the same value.

Sines

In a right triangle ABC , the ratio of the side opposite $\angle A$ to the hypotenuse of the triangle is called the **sine** of angle A and is written $\sin A$. Thus, in either figure on page 299,

$$\sin A = \frac{BC \text{ (the side opposite } \angle A \text{)}}{AB \text{ (the hypotenuse)}}$$

An approximate value for the sine of an angle may be read from a figure drawn on squared paper, as at the right. Take $OA = 10$ spaces. Construct the arc AF of a circle. From O , with the aid of a protractor draw radii making angles with OA , as indicated. Since each radius is equal to 10, the sine of each angle may be read directly from the length of the perpendicular line. For example:



$$\sin 25^\circ (\angle BOG) = \frac{BG}{OB} = \frac{4.2}{10} = .42$$

From the figure you can easily see that the value of the sine of an angle depends on the number of degrees in the angle. Hence the sine of an angle is a **function** of the angle.

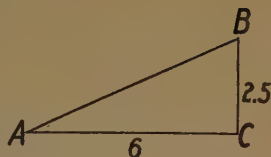
EXERCISES

Construct a figure as in the diagram above with radii that make angles of 35° , 50° , and 75° with OA . From the figure estimate the sines of the following angles:

1. 35° 2. 50° 3. 75°

4. Construct right triangle ABC with $AB = 10$ in., $AC = 8$ in., and $BC = 6$ in. What is the value of $\sin A$? of $\sin B$?

5. If in the right triangle at the right, $AC = 6$ and $BC = 2.5$, what is the value of AB ? (See page 293.)



6. In the triangle of ex. 5, find $\sin A$.

7. In the same figure, what is $\sin B$?

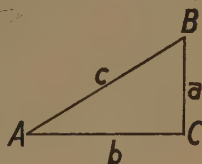
8. Draw an isosceles right triangle having an arm equal to 3 in. What is the value of each of the acute angles? What is the sine of one of the acute angles? How accurately did you measure the 3-inch arm?

9. Draw a $30^\circ-60^\circ$ right triangle with the side opposite the 30° angle equal to $1\frac{1}{2}$ in. Find the value of $\sin 30^\circ$ and of $\sin 60^\circ$. Express the result in simplest radical form. (See page 240.)

Cosines and Tangents

Besides the sine of an angle, there are two other ratios that are in frequent use.

The **cosine** of an angle is the ratio of the side adjacent to the angle to the hypotenuse of the triangle. "Cosine of angle A " is written $\cos A$. In the figure at the right,



$$\cos A = \frac{AC \text{ (the side adjacent to } \angle A \text{)}}{AB \text{ (the hypotenuse)}}, \text{ or } \cos A = \frac{b}{c}$$

The **tangent** of an angle is the ratio of the side opposite the angle to the side adjacent to the angle. "Tangent of angle A " is written $\tan A$. In the figure above,

$$\tan A = \frac{BC \text{ (the side opposite } \angle A \text{)}}{AC \text{ (the side adjacent to } \angle A \text{)}}, \text{ or } \tan A = \frac{a}{b}$$

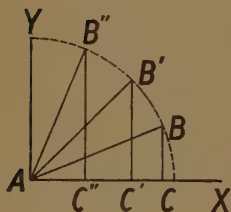
EXERCISE. In the figure above, if $a = 6$ and $b = 8$ find:

(1) $\sin A$; $\cos A$; $\tan A$.

(2) $\sin B$; $\cos B$; $\tan B$.

An interesting connection between the sine, the cosine, and the tangent of an angle and one that sometimes may be used in checking your work is shown below :

$$\frac{\sin A}{\cos A} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a}{c} \times \frac{c}{b} = \frac{a}{b} = \tan A \quad (\text{See figure on page 299.})$$



As angle A changes in size from 0° to 90° :

$\sin A$ increases from 0 to 1.

$\cos A$ decreases from 1 to 0.

$\tan A$ increases from 0 without limit.

The sine, the cosine, and the tangent of an angle are the three principal **functions** of an angle studied in trigonometry.

EXERCISES

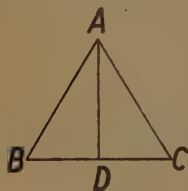
Give decimal results correct to two places.

1. Construct a right triangle ABC ($\angle C = 90^\circ$) with $AC = 4.5$ in. and $BC = 6$ in. Find AB .

2. In the triangle of ex. 1 find :

(a) $\cos A$ (b) $\cos B$ (c) $\tan A$ (d) $\tan B$

3. Draw an isosceles right triangle. From it find $\cos 45^\circ$ and $\tan 45^\circ$. How does the value of $\cos 45^\circ$ compare with that of $\sin 45^\circ$?



4. Draw an equilateral triangle and an altitude, as in the diagram. What is the value of each angle of $\triangle ABC$? What is the value of each angle of $\triangle ABD$? Explain.

5. (a) In the figure for ex. 4, if $AB = 2$ in., what is the length of BD ? What is the length of AD ? Explain.

(b) Write the value of each of the following ratios in simplest radical form :

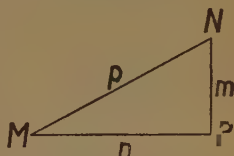
$$\cos 30^\circ \quad \cos 60^\circ \quad \tan 30^\circ \quad \tan 60^\circ$$

6. Using the results obtained in ex. 8 and 9, page 299, and ex. 3 and 5, page 300, complete the following table with values stated in simplest radical form :

Size of angle	30°	45°	60°
Sine			
Cosine			
Tangent			

7. What relation is there between $\sin 30^\circ$ and $\cos 60^\circ$? between $\cos 30^\circ$ and $\sin 60^\circ$? What name is applied to the angles 30° and 60° when taken together?

8. Draw any right triangle, as $\triangle MNP$ at the right. What relation exists between $\angle M$ and $\angle N$? (See ex. 1, page 287.) between $\sin M$ and $\cos N$? What conclusion can you draw about the sine of an angle and the cosine of its complement?



9. In the figure for ex. 8, state $\tan M$; $\tan N$. What relation exists between the tangent of an angle and the tangent of the complementary angle?

10. If $\angle M$ increases in size, what change takes place in $\tan M$? in $\sin M$? in $\cos M$?

11. If $\angle M$ in triangle MNP increases in size, what change takes place in $\angle N$?

12. If $\angle N$ decreases in size, what change takes place in $\tan N$? in $\sin N$? in $\cos N$?

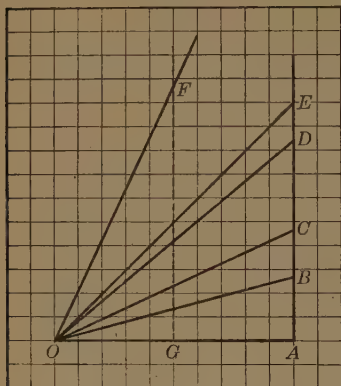
13. What angle function was used in working ex. 4, page 109?

14. In the figure for ex. 4, page 109, find :

(a) $\sin A$ (b) $\cos A$ (c) $\tan A$

15. In the figure for ex. 4, page 109, find $\tan C$.

16. Make a drawing on squared paper, similar to the figure below. Explain how the approximate values of tangents of angles may be read directly from the drawing.



17. If $\angle BOA = 15^\circ$, find $\tan 15^\circ$.

18. If $\angle COA = 25^\circ$, find $\tan 25^\circ$.

19. If $\angle DOA = 40^\circ$, find $\tan 40^\circ$.

20. If $\angle EOA = 45^\circ$, find $\tan 45^\circ$.

21. If $\angle FOA = 65^\circ$, find $\tan 65^\circ$.

22. Construct on squared paper a drawing similar to that on page 298 and explain how approximate values of the cosines of angles may be read from the drawing.

23. From the drawing of ex. 22, find the approximate values of :

(a) $\cos 15^\circ$

(b) $\cos 40^\circ$

(c) $\cos 55^\circ$

Compare your results with those you obtained for ex. 1, 2, and 3, page 298.

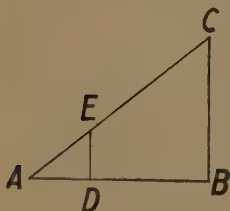
24. Explain why the sine, the cosine, and the tangent of an angle are called functions of the angle.

25. In the figure, $ED \perp AB$ and $CB \perp AB$. If $AD = 7.5$, $ED = 10$, and $AB = 18$, find: (a) AE ; (b) BC ; (c) AC .

26. Show that $\sin \angle AED = \sin C$.

27. Show that $\sin A$ in $\triangle AED = \sin A$ in $\triangle ACB$.

28. Find: (a) $\tan \angle E$ in $\triangle AED$; (b) $\tan C$.

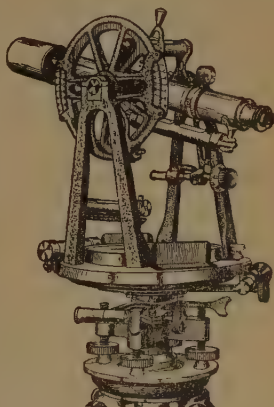


The Transit

In measuring land, it is generally easier to measure the size of an angle than the length of a line. For this purpose, surveyors and engineers use an instrument called a **transit**. This consists of a telescope and two graduated circles that serve as protractors. One of the protractors is vertical, for measuring heights and depressions. The other is horizontal, for measuring horizontal angles.

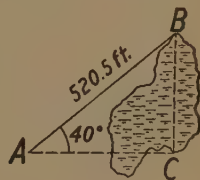
To measure an angle, the telescope is first pointed along one side of the angle and the corresponding reading on the protractor is noted. Then the telescope is pointed along the other side of the angle and the protractor reading is noted. The difference between the two readings gives the required angle.

When measuring the height of objects, allowance must always be made for the height of the telescope above the ground.



Approximate Numbers

A surveyor wishes to find the distance across a marsh which it is difficult to measure. At point C he sights a point B across the marsh. Then he turns the telescope of the transit through an angle of 90° and sights point A so that $\angle ACB$ is 90° . He is able to measure the line AB and finds that it is 520.5 ft. He measures $\angle A$ and finds that it is 40° .



When the surveyor measures the distance AB and finds that it is 520.5 ft., he cannot be sure that it is exactly 520.5 ft. and not 520.47 ft. or 520.52 ft. He is sure that the distance is correct to the nearest tenth of a foot; but his measuring tape and the lay of the ground will not permit him to make more accurate measurements. When he measures shorter distances, he may get the result correct to the nearest hundredth of an inch or thousandth of an inch, if his instruments are of sufficient accuracy and he is careful in taking the measurements.

All measured numbers are only *approximately* correct. They may be a trifle too much or too little. In problems connected with measured numbers, the results are correct only to the number of figures that the measurements are correct.

EXAMPLE. Find the distance BC in the diagram at the right.

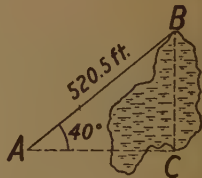
$$\text{SOLUTION. } \sin A = \frac{BC}{AB}, \text{ or } \sin 40^\circ = \frac{BC}{520.5}$$

$$BC = 520.5 \sin 40^\circ$$

The table of sines (page 307) shows that

$$\sin 40^\circ = .6428$$

$$\begin{aligned} \text{Hence } BC &= 520.5 \times .6428 \\ &= 334.5774 \quad \text{Ans. } 334.6 \text{ ft.} \end{aligned}$$



The table of sines is correct to four decimal places and the measured distance is correct to four figures, hence four figures are retained in the result. Since, however, the fifth figure in the product is 7, it is dropped and the fourth figure is changed to 6.

The figures that are retained in the answer to the problem are called **significant figures**. They signify, or tell, the accuracy of the number.

Thus, if you count your money and have exactly \$100, the two zeros are significant. If you measure a distance and find that it is 37.8 ft., the three figures are significant. If you express this distance to the nearest foot, you state it, with two significant figures, as 38 ft. The same distance may be expressed with one significant figure as 40 ft.

Table of Trigonometric Ratios

On page 307 is a table of the values of the trigonometric ratios of angles from 1° to 90° . The approximate value of the ratios for all angles can be found from this table. More elaborate tables are used for practical work where greater accuracy of results is required.

EXAMPLE 1. Find $\sin 53^\circ 20'$.

SOLUTION. $\sin 53^\circ 20'$ lies between $\sin 53^\circ$ and $\sin 54^\circ$. Since there are 60' in a degree, 20' equals $\frac{1}{3}$ of a degree.

From the table, $\sin 54^\circ = .8090$

$\sin 53^\circ = .7986$

Difference = .0104

$\frac{1}{3}$ of .0104 = $.0034\frac{2}{3}$, or .0035

Hence $\sin 53^\circ 20' = .7986 + .0035 = .8021$

The process by which the trigonometric ratio of an angle not given in the table is obtained is called **interpolation**. It is based on the principle that the ratio varies just as the angle does if the function is an increasing one or inversely as the angle does if the function is a decreasing one. While this is not exactly true, the method gives a very good approximate result.

EXAMPLE 2. What is $\cos 23^\circ 50'$?

SOLUTION. $\cos 23^\circ 50' = \cos 23\frac{5}{6}^\circ$, or $\cos 23.83^\circ$

From the table, $\cos 23^\circ = .9205$

$\cos 24^\circ = .9135$

Difference = .0070

.83 of .0070 = .0058

Since the cosine of an angle *decreases* as the angle increases, the proportional part of the difference (.0058) is to be *subtracted* from the value of $\cos 23^\circ$.

Hence $\cos 23^\circ 50' = .9205 - .0058 = .9147$

EXERCISES

From the table on page 307, find :

- | | | |
|--------------------|--------------------|--------------------|
| 1. $\sin 22^\circ$ | 4. $\cos 38^\circ$ | 7. $\tan 50^\circ$ |
| 2. $\sin 48^\circ$ | 5. $\cos 72^\circ$ | 8. $\tan 20^\circ$ |
| 3. $\sin 70^\circ$ | 6. $\cos 85^\circ$ | 9. $\tan 34^\circ$ |

Using interpolation, find :

- | | | |
|-------------------------|-------------------------|-------------------------|
| 10. $\sin 35^\circ 20'$ | 13. $\cos 47^\circ 10'$ | 16. $\tan 75^\circ 30'$ |
| 11. $\sin 43^\circ 40'$ | 14. $\cos 81^\circ 30'$ | 17. $\tan 60^\circ 45'$ |
| 12. $\sin 58^\circ 27'$ | 15. $\cos 23^\circ 47'$ | 18. $\tan 25^\circ 32'$ |

Find the value of angle A when :

- | | | |
|----------------------|----------------------|-----------------------|
| 19. $\sin A = .2588$ | 22. $\cos A = .9703$ | 25. $\tan A = 3.7321$ |
| 20. $\sin A = .5878$ | 23. $\cos A = .8988$ | 26. $\tan A = 1.6003$ |
| 21. $\sin A = .9703$ | 24. $\cos A = .1045$ | 27. $\tan A = .2309$ |

28. Find angle A when $\sin A = .6734$.

SOLUTION. $\sin A = .6734$ is not found in the table of sines, but it is between $\sin 42^\circ = .6691$ and $\sin 43^\circ = .6820$. Use interpolation to find A .

$$\sin 43^\circ = .6820$$

$$\sin A = .6734$$

$$\sin 42^\circ = .6691$$

$$\sin 42^\circ = .6691$$

$$\text{Difference} = .0129$$

$$\text{Difference} = .0043$$

Since A lies between 42° and 43° , assume that the difference is $\frac{.0043}{.0129}$ of 1° , or $\frac{43}{129} \times 60' = 20'$.

Hence $\angle A = 42^\circ 20'$.

Find the value of angle A when :

- | | | |
|----------------------|----------------------|-----------------------|
| 29. $\sin A = .6967$ | 32. $\cos A = .0640$ | 35. $\tan A = .5619$ |
| 30. $\sin A = .6134$ | 33. $\cos A = .3007$ | 36. $\tan A = .2742$ |
| 31. $\sin A = .8962$ | 34. $\cos A = .9588$ | 37. $\tan A = 3.1716$ |

38. If the arms of a right triangle are 3 in. and 4 in., respectively, what is the value of each of the two acute angles?

HINT. The tangent of one angle will be $\frac{3}{4}$, or .7500.

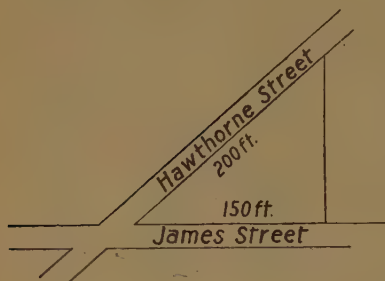
TABLE OF SINES, COSINES, AND TANGENTS

ANGLE	SINE	COSINE	TANGENT	ANGLE	SINE	COSINE	TANGENT
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	infinite

EXERCISES

In the following exercises, find the angles correct to the nearest minute.

1. (a) A city lot is at the intersection of the two streets shown in the drawing. If the back line is at right angles with James Street, what is the angle of intersection of the streets?



(b) In two different ways, find the length of the back line. How do the two results compare?

HINT. (a) If A is the angle of intersection, $\cos A = \frac{150}{200} = .7500$. From the table, $\angle A = 41^\circ 24'$. Show that this is true.

(b) If L is the back line, $\frac{L}{200} = \sin 41^\circ 24'$, or $L = 200 \sin 41^\circ 24'$. Show that $L = 132.3$.

By means of the Pythagorean principle, $L = \sqrt{200^2 - 150^2} = 50\sqrt{7}$, or $L = 132.28$, or $L = 132.3$.

2. Give the equation necessary to find a , b , or c in each of the following figures. Show how the result may be checked in each case.

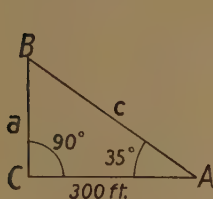


FIG. 1

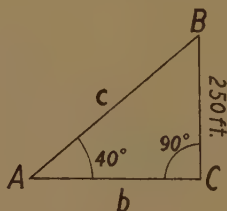


FIG. 2

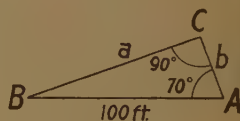


FIG. 3

3. Find the value of the parts not given in the triangles of ex. 2. Check each result.

Find the lines and the angles of triangle ABC that are missing in the following exercises. Angle C is the right angle. Use the Pythagorean theorem only as a check:

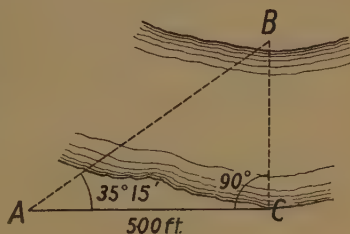
	a	b	c	Angle A	Angle B
4.	500			25°	
5.		320		$32^\circ 30'$	
6.			458	$38^\circ 20'$	
7.		378			43°
8.	430				$53^\circ 25'$
9.	208			$32^\circ 20'$	
10.		1050			$48^\circ 40'$
11.	360	420			
12.	250		650		
13.		88	110		
14.	240	450			

15. A ship sails northwest a distance of 70 mi. If the shore line extends north and south, how far is the ship from land?

HINT. A northwest line makes an angle of 45° with a west line.

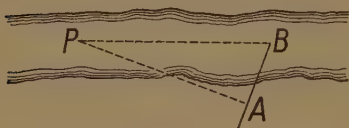
16. An engineer in surveying for a bridge made measurements indicated in the figure at the right to find the distance across the river. What is the width of the river?

17. In ex. 16, what is the distance from A to B ?

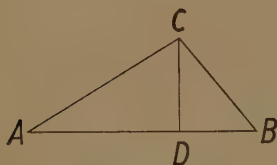


18. An enemy battery was hidden behind a range of hills. To find the approximate distance to the battery, a line 2000 ft. long was laid off at right angles to the line of fire. At the farther end of this line the apparent position of the battery was sighted. This made an angle of $80^\circ 15'$ with the base line. Draw a figure to represent these conditions and find the approximate distance of the battery from the line of defense.

19. The boys of a scout camp wish to lay out a mile rowing course on a lake. They lay out a base line BA of indefinite length. The course is to run in the direction of BP that makes an angle of 70° with the base line BA . Find the point A from which the point P may be sighted if angle A is a right angle.



20. From the data given in ex. 19, find the distance AP .



21. Find the altitude of triangle ABC if it is known that $AB = 6$ ft., $AD = 4$ ft., $\angle A = 32^\circ 20'$, and $\angle C = 90^\circ$. Check.

22. What is the area of the triangle in ex. 21?

23. If in the triangle in ex. 21, $CB = 24$ in., $\angle C = 90^\circ$, and $\angle B = 53^\circ 35'$, what is the length of the altitude, CD ? of AB ? Find the area of the triangle.

24. In the figure for ex. 21, if $AC = 22.5$ ft. and $CB = 12$ ft., find: (a) $\angle A$; (b) $\angle B$; (c) AD ; (d) DB ; (e) CD .

25. In two different ways, find the area of the triangle in ex. 24.

26. If a school football field is 300 ft. by 150 ft., find the angle that a diagonal line from one corner to the opposite corner makes with the goal line.

27. Find the length of the diagonal line in ex. 26 by using the angle the line makes with the goal line. Check.

Solve the following triangles ($\angle C = 90^\circ$); that is, find the sides and the angles not given, and the area:

28. $A = 52^\circ 30'$, $c = 30$ in.

30. $A = 48^\circ 36'$, $a = 120$ ft.

29. $A = 30^\circ 25'$, $b = 42$ in.

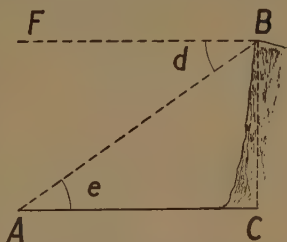
31. $a = 75$ ft., $b = 63$ ft.

32. Two girl scouts make a camping tent in the form of a square pyramid. The ground line of each side is 10 ft. and the slant corner lines make an angle of 60° with the ground line. If no allowance is made for waste, find the number of square yards of cloth necessary to make the tent. Draw a diagram to represent the tent.

33. Find the total area of a tent similar to that of ex. 32 if the ground line of each side is 12 ft. and one of the slant corner lines is 15 ft.

The **angle of elevation** of an object is illustrated at the right. It is the angle that the line of sight from the observer to the object makes with the horizontal line from the point of observation.

A similar definition holds for the **angle of depression** of an object. In the first case the object is *above* the point of observation, while in the second case it is *below* the point of observation.

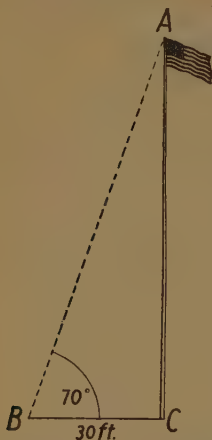


Thus, in the drawing, $\angle e$ is the angle of elevation of point B above the observation point A , and $\angle d$ is the angle of depression of object A below the point of observation B . $\angle e = \angle d$.

When the height of objects is measured or the distance of the depression is found, the elevation of the observer above the ground must be considered. In the following problems, consider only the elevation above the eye of the observer or the depression below the eye.

EXERCISES

1. Frank wishes to find the height of the flagpole in the school yard. He measures a distance of 30 ft. from the base of the pole and then measures the angle of elevation of the top of the pole as 70° . How high is the pole?



2. At the time when Frank makes the measurements of ex. 1, the length of the shadow of the pole is 40 ft. What is the angle of elevation of the sun?

3. A ladder 30 ft. long rests against a building so as to make an angle of 75° with the ground. How far from the ground is the top of the ladder?

4. If a ladder placed as in ex. 3 makes an angle of 78° with the ground at a distance of 6 ft. from the building, find the length of the ladder. How high on the building does it reach?

5. Find the approximate height of the Eiffel Tower if the angle of elevation of the top of the tower, when measured 200 ft. from the base, is $78^\circ 40'$.

6. Find the approximate height of the Washington Monument if the angle of elevation of the sun is $61^\circ 36'$ at a time when the shadow cast by the monument is 300 ft. long.

7. From the top of a lighthouse 70 ft. high the angle of depression of a boat is $12^\circ 42'$. How far from the lighthouse is the boat?

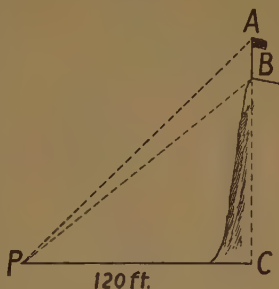
8. An officer of a boat observes that the angle of elevation of the top of a lighthouse is $10^\circ 15'$. If the top of the lighthouse is 120 ft. above the water, how far from its base is the boat?

9. The grade of a road that rises 10 ft. for each 100 ft. measured along the road is said to be a 10% grade. Find the angle of elevation of the road.

10. In some places the railroad up Pikes Peak has an angle of elevation of about $10^{\circ} 12'$. What per cent grade does this represent? How high does such a railroad rise in 200 ft. of track?

11. At a point P , 120 ft. from the base of a cliff, the angle of elevation of the base of a flagpole is observed to be $39^{\circ} 50'$ and that of the top of the flagpole is seen to be 45° . Find the height of the pole.

HINT. First find AC , then BC . Then find their difference.



12. An airplane is above a point 1 mi. distant from the observer. If the angle of elevation of the airplane is $42^{\circ} 27'$, how high is it above the ground?

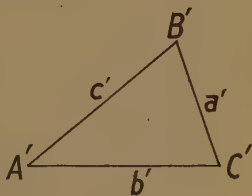
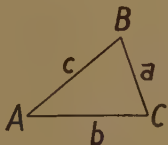
13. A telephone pole 30 ft. high is strengthened by a guy wire that has an elevation of $72^{\circ} 48'$. Find the length of the guy wire.

14. While flying a kite, Ray noticed that the angle of elevation of the kite was $44^{\circ} 24'$. If the kite string was 180 ft. long, how high was the kite?

REVIEW EXERCISES

1. Similar triangles are triangles that have —.

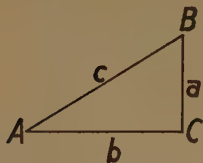
2. If in the diagram at the right $\triangle ABC$ is similar to $\triangle A'B'C'$, then $\frac{a}{?} = \frac{b}{?} = \frac{c}{?}$.



3. If $\angle A = 40^{\circ}$, $\angle B = 70^{\circ}$, then $\angle C = ?$
 $\angle A' = ?$ $\angle B' = ?$ $\angle C' = ?$

4. If in the triangles of ex. 2, page 313, $a = 2$ in., $b = 2.5$ in., $c = 2.5$ in., and $a' = 6$ in., then $b' = ?$ $c' = ?$

5. If a triangle is formed by increasing each side of $\triangle ABC$ by 5 in., will the new triangle be similar to $\triangle ABC$? Why?



6. If a triangle is formed by lines each 5 times as long as the sides of $\triangle ABC$, will the new triangle be similar to $\triangle ABC$? Why?

7. The three ratios that are used in trigonometry between the sides of a right triangle are —, —, and —.

8. The sine of an angle is the ratio of — to the —.

9. The value of the sine of an angle depends on —.

10. The cosine of an angle is the ratio of —.

11. The tangent of an angle is the ratio of —.

12. If in the figure above the lengths of the sides of $\angle A$ increase and $\angle C$ remains a right angle, then $\sin A$ —, $\cos A$ —, and $\tan A$ —.

13. The sine, the cosine, and the tangent of an angle are called — of an angle because —.

14. If a and c are known, b may be found by —.

15. If a and b are known, $\angle A$ may be found by —.

16. If c and $\angle B$ are known, a may be found by —.

17. If b and $\angle A$ are known, c may be found by —.

18. The sine of $\angle A =$ the — of $\angle B$.

19. The sine of any acute angle equals the — of its complement.

20. The angle of elevation of an object is the angle —.

21. The angle of depression of an object is —.

22. Measured numbers are approximate because —.

23. The term *interpolate* means —.

If $S = 4\pi r^2$, find S when $r = 3\frac{1}{4}$ and $r = 3.5$



CHAPTER XVII

ALL SORTS OF EXERCISES



Addition

Find the indicated sums:

- | | | | |
|--|--|--|--|
| 1. $- 9$
13
28
$- 15$
<hr style="width: 100%;"/> | 5. 33
$- 5$
28
17
<hr style="width: 100%;"/> | 9. $25 a^2$
$12 a^2$
$- 17 a^2$
$19 a^2$
<hr style="width: 100%;"/> | 13. $4.5 a$
$- 13.2 a$
$12.3 a$
a
<hr style="width: 100%;"/> |
| 2. $- 17$
$- 28$
$- 52$
98
<hr style="width: 100%;"/> | 6. 4
25
13
$- 29$
<hr style="width: 100%;"/> | 10. $52 ax$
$73 ax$
$- 65 ax$
$- 12 ax$
<hr style="width: 100%;"/> | 14. $- 7.2 bx$
$- 8.8 bx$
$2.7 bx$
$- 3.4 bx$
<hr style="width: 100%;"/> |
| 3. 15
$- 23$
19
$- 10$
<hr style="width: 100%;"/> | 7. 15
$- 3$
9
$- 13$
<hr style="width: 100%;"/> | 11. $3 y$
$- 12 y$
$- 10 y$
$18 y$
<hr style="width: 100%;"/> | 15. $3.1 r$
$- 2.9 r$
$5.8 r$
$- 6.4 r$
<hr style="width: 100%;"/> |
| 4. 36
72
$- 91$
25
<hr style="width: 100%;"/> | 8. $- 42$
15
73
$- 21$
<hr style="width: 100%;"/> | 12. $- 53 x$
$30 x$
$- 2 x$
$15 x$
<hr style="width: 100%;"/> | 16. $- 3.7 m^2$
$4.1 m^2$
$.2 m^2$
$- 3.5 m^2$
<hr style="width: 100%;"/> |
| 17. $- ax^2 + 3 bx - 2 c$
$3 ax^2 - 5 bx + 3 c$
<hr style="width: 100%;"/> | | 19. $3 m^2 - 2 m + 5$
$- 8 m^2 + 5 m - 7$
<hr style="width: 100%;"/> | |
| 18. $4 ax^2 - bx + b^2$
$- 7 ax^2 + 5 bx - 3 b^2$
$8 ax^2 - 3 bx + 2 b^2$
<hr style="width: 100%;"/> | | 20. $3 m^2 - 8 m + 17$
$2 m^2 + 7 m - 15$
$- m^2 + 5 m + 3$
<hr style="width: 100%;"/> | |

Find the sum of :

21. $3x^2 + 5x - 7$, $2x + 13x^2 - 8$, and $15x^2 - 12x + 3$
 22. $5ax + 3by - c$, $7by - 8ax + 5c$, and $9ax - 10by$
 23. $3(a + b) + 5(c - d)$, $4(a + b) - 6(c - d)$, and $7(c - d)$
 24. $7(2x - y) - 3(x + y)$, $5(x + y) + 4(2x - y)$, and $(2x - y) + (x + y)$
 25. $x^2 + xy - y^2$, $2xy - x^2 + 3y^2$, and $4x^2 + 5y^2 - 3xy$
 26. $3x^2 + xy$, $7y^2 - 2x^2$, $8x^2 + 7xy - 5y^2$, and $-3xy$.
 27. How many x 's are there in $ax + bx - x$?
 28. How many y 's are there in $2ay - 3by + 5y$?
 29. Express $8 \times 17 + 12 \times 17 - 3 \times 17 + 9 \times 17$ as a multiple of 17.
 30. Express $15 \times 23 + 9 \times 23 - 25 \times 23 + 6 \times 23$ as a multiple of 23.

Subtraction

Subtract the lower number from the upper one :

- | | | | |
|--|---|---|--|
| 1. $\begin{array}{r} 5 \\ - 2 \\ \hline \end{array}$ | 5. $\begin{array}{r} 13 \\ 25 \\ \hline \end{array}$ | 9. $\begin{array}{r} 12a \\ - a \\ \hline \end{array}$ | 13. $\begin{array}{r} xy \\ - 7xy \\ \hline \end{array}$ |
| 2. $\begin{array}{r} -18 \\ 3 \\ \hline \end{array}$ | 6. $\begin{array}{r} 53 \\ - 12 \\ \hline \end{array}$ | 10. $\begin{array}{r} 23a^2 \\ 12a^2 \\ \hline \end{array}$ | 14. $\begin{array}{r} 2ax \\ 5ax \\ \hline \end{array}$ |
| 3. $\begin{array}{r} 15 \\ - 22 \\ \hline \end{array}$ | 7. $\begin{array}{r} 28 \\ - 32 \\ \hline \end{array}$ | 11. $\begin{array}{r} 15x \\ - 12x \\ \hline \end{array}$ | 15. $\begin{array}{r} 17ab \\ - ab \\ \hline \end{array}$ |
| 4. $\begin{array}{r} -17 \\ - 15 \\ \hline \end{array}$ | 8. $\begin{array}{r} 19 \\ 21 \\ \hline \end{array}$ | 12. $\begin{array}{r} -7x^2 \\ - 17x^2 \\ \hline \end{array}$ | 16. $\begin{array}{r} 3pq \\ - 12pq \\ \hline \end{array}$ |
| 17. $\begin{array}{r} 2a \div 3b \\ 4a + 5b \\ \hline \end{array}$ | 19. $\begin{array}{r} 2a + 5b - 7 \\ a - 7b + 3 \\ \hline \end{array}$ | | |
| 18. $\begin{array}{r} 7ax + 5 \\ - 2ax - 7 \\ \hline \end{array}$ | 20. $\begin{array}{r} 3x^2 + 5x + 2 \\ - 8x^2 - 7x + 3 \\ \hline \end{array}$ | | |

21. What number added to $3x + 7y - 8z$ will give $5x + 2y + z$?

22. What number added to $15ax + 13by - 7$ will give $2ax - by$?

23. What number subtracted from $3x^2 + 5x - 2$ will give $8x + 3$?

24. What number subtracted from $6x^2 - 7xy + 8y^2$ will give $15xy$?

25. From the sum of $6x + 7y - 8z$ and $2x - 7y + 3z$ subtract $3x + y - 2z$.

26. Subtract the difference between $7a^2 - 2ab + b^2$ and $3a^2 + 5b^2$ from $8a^2 + 3ab$.

27. By how much does $2a + b - c$ exceed $5b + c$?

28. By how much must $3x^2 + x - 7$ be increased so that the result shall be $5x^2 - 2x + 19$?

29. What is the excess of $3x^2 - 7xy + y^2$ over $x^2 + 10xy - 8y^2$?

30. $7ax - 2bx + 5$ exceeds $ax + 3bx - 7$ by what expression?

31. To the sum of $15x^2 - 7xy + 8y^2$ and $x^2 + 2xy - 13y^2$ add the difference between $3x^2 - 5xy + y^2$ and $2x^2 + 8xy - 5y^2$.

32. By how much must $ax + by - cz$ be increased so that the result shall equal the difference between $5ax - 2by + 3cz$ and $2ax + by - 13cz$?

33. Find the sum of $3a^2x^2 - 7ax + 5$, $4a^2x^2 + 3ax + 11$, and $27ax - 17 - 5a^2x^2$.

34. Find the difference between $7a^2x^2 - 15ax + 1$ and $a^2x^2 + 3ax$.

35. By how much does the answer to ex. 34 exceed the answer to ex. 33?

Numerical Substitution

If $a = 3$, $b = -2$, $c = -1$, $x = -3$, and $y = 0$, find the value of:

1. $2a + 3b - b^2$

2. $2a^2 - b + b^3$

3. $3a + bc - cx$

4. $2x^2 + ay + x^3$

5. $ab + bc + ca$

6. $ax - by - ab$

7. $2a + 3b + xy$

8. $a^2 - 2a - 15$

9. $a^2 + b^2 + c^2$

10. $a^2b + ab^2 - cx$

11. $a(b + c - 5x)$

12. $(a + b)(2c + x)$

13. $x^2 - 3(x - y) + 4(x - 2y)$

14. $x^3 - 5x^2 + 8x - 4$

15. Find the value of $-b + \sqrt{b^2 - 4ac}$ if $a = 3$, $b = 9$, and $c = -30$; if $a = 2$, $b = -11$, and $c = 14$.

16. Find the value of $-b - \sqrt{b^2 - 4ac}$ if $a = 3$, $b = 13$, and $c = -10$; if $a = 3$, $b = -14$, and $c = -5$.

17. If $C = \frac{5}{9}(F - 32)$, find C when $F = 75^\circ$; when $F = -8^\circ$.

18. If $F = \frac{9}{5}C + 32$, find F when $C = 20^\circ$; when $C = -10^\circ$.

19. If $C = \frac{E}{R + r}$, find C when $E = 110$, $R = 4.3$, and $r = 2.8$.

20. If $S = \frac{rl - a}{r - l}$, find S when $a = 2.5$, $r = 2$, and $l = 80$.

21. Find A if $A = p(1 + rt)$ and $p = 580$, $r = 4\frac{1}{2}\%$, $t = 3$.

22. Find A if $A = p(1 + r)^t$ and $p = 200$, $r = .05$, $t = 3$.

23. Find S if $S = \frac{a}{2}(2t - 1)$ and $a = 4.9$, $t = 5$.

24. If $S = \frac{1}{2}at^2 + vt$, find S when $a = 32.16$, $t = 4$, and $v = 20$.

25. If $S = \frac{n}{2}(a + l)$, find S when $n = 12$, $a = 5$, and $l = 38$.

26. Find T if $T = 2\pi r(h + r)$ and $\pi = 3.14$, $r = 5.1$, $h = 8$.

27. If $S = 2\pi rh$, find S when $\pi = 3.14$, $r = 4.2$, and $h = 4.7$.

28. If $S = 4\pi r^2$, find S when $\pi = 3.14$ and $r = 3.5$.

Parentheses

Remove the parentheses and unite like terms :

1. $2x - (3x + 5)$
2. $8x - (4x - 2)$
3. $7a + (3x - 2a)$
4. $(6a + 3) - 2a$
5. $(-5a + 7) + 4a$
6. $7a - (-2a + 5)$
7. $8x - 3(5 - 2x)$
8. $9a - 3a - (5 - 2a)$
9. $3y + 2a - 3(y - 2a)$
10. $5xy - y(2x + 3)$
11. $22 - 3(2x - 4) + (9x + 3)$
12. $7a - [(5a - 3) + 3a]$
13. $3a + [5a - (9a - 4) + 7a]$
14. $2x^2 - [3x^3 + (4x - 2x^3) - x^3] - 4x^2$
15. $3x - 5 - [2x + (8 - 12x) + 3] - 7x$

Group the last three terms in a parenthesis :

16. $49b^2 - x^2 - 2xy - y^2$
17. $16a^2 - 25x^2 + 10xy - y^2$
18. $25y^2 - 16x^2 - 48ax - 36a^2$
19. $36m^2 + 28ab - 49a^2 - 4b^2$
20. $9x^2 - 9 + 30a - 25a^2$

Group the terms containing x or y in one parenthesis and the other terms in another parenthesis :

21. $a^2 - x^2 + 2xy - y^2$
22. $a^2 - x^2 - 4y^2 + 4xy$
23. $25x^2 + 9y^2 - 36a^2 - 30xy$
24. $x^2 + y^2 - 16a^2 - 2xy$
25. $x^2 - a^2 - b^2 + 9y^2 - 6xy + 2ab$

Write as a single term :

26. $12\sqrt{2} - 2d\sqrt{2} + 18\sqrt{2} - 5d\sqrt{2}$
27. $2\sqrt{3} + (5a - 2)\sqrt{3} - \sqrt{3} + a\sqrt{3}$
28. $(5 - 7a)\sqrt{2} - 3(a - 2)\sqrt{2} + \sqrt{2}$
29. $a\sqrt{a - x} - (2 - x)\sqrt{a - x} + \sqrt{a - x}$
30. $ax + bx - 7x + (a - b)x$

Multiplication

Find the indicated products :

$$\begin{array}{r} 1. \quad 3x - 2 \\ \quad x + 5 \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad y^2 - 3a \\ \quad y^2 + 7a \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad ax + b \\ \quad ax - c \\ \hline \end{array}$$

$$\begin{array}{r} 2. \quad x + 7a \\ \quad 2x - 3a \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad ab - c^2 \\ \quad ab + 5c^2 \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad 7x^2 - ab \\ \quad 3x^2 + c \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 7x - 2y \\ \quad -2x + 5y \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad ax^2 + b \\ \quad x^2 - b \\ \hline \end{array}$$

$$\begin{array}{r} 11. \quad nx^2 - 8 \\ \quad 2nx^2 + .5 \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad 3a + b \\ \quad 2a - 5b \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad 2a^3 + b \\ \quad 3a^3 - b \\ \hline \end{array}$$

$$\begin{array}{r} 12. \quad 5a^2x - 1.2 \\ \quad 3a^2x + 2.5 \\ \hline \end{array}$$

Simplify the following :

$$13. (a + x)^2 - 5(2a - x)(a + 3x)$$

$$14. (2x - y)(4x^2 + 2xy + y^2) + 8x(x - 2y)^2$$

$$15. (3x - y)^3 + 2x(x + 3y)^2$$

$$16. (2a - b)^2 - (3a + b)^2 + (a - 2b)^2$$

$$17. (a^4 - a^3 + a^2 - 1)(a + 1)$$

$$18. (x^2 - 2x + 2)(x^2 + 2x + 2)$$

$$19. (a^2 - 3ab + 2b^2)(3ab - 2b^2)$$

$$20. (2x^3 + 3x^2 - 5)(3x - 4)$$

$$21. (a^2 + 2a + 1)a^n$$

$$22. (a^{2n} - 2a^n + 4)(a^n + 2)$$

23. If $(4r - 5)(2r + 3) = 8r^2 + cr - 15$, what must be the value of c ?

24. If $(2x - 5)(3x + 2) = 6x^2 - bx + c$, find the values of b and c .

25. If $(ax + 4)(8x - 3) = 56x^2 + 11x - 12$, find the value of a .

26. If $(2x + p)(7x + 2) = 14x^2 - 17x + 2p$, find the value of p .

Division

Find the indicated quotients. Check ex. 1 to 5 by substituting 2 for x . Check ex. 6 to 15 by multiplication.

1. $(6x^2 + x - 35) \div (3x - 7)$
2. $(21x^2 + 35x - 40) \div (7x - 5)$
3. $(10x^2 + 3x - 27) \div (2x - 3)$
4. $(10x^2 + 31x + 20) \div (2x + 5)$
5. $(2x^2 - 11x - 28) \div (x - 7)$
6. $(6x^3 - 7x^2 + 14x - 8) \div (2x^2 - x + 4)$
7. $(x^3 - 5x^2 + 8x - 4) \div (x - 2)$
8. $(6x^3 - 23x^2 + 29x - 12) \div (2x - 3)$
9. $(a^4 + 2a^3 + 26a - 6a^2 - 15) \div (a^2 + 4a - 3)$
10. $(12x^4 - x^3 - 3x + 10 - 27x^2) \div (4x^2 + x - 2)$
11. $(x^4 - 5x^3 + 15x^2 - 22x + 20) \div (x^2 - 2x + 4)$
12. $(a^4 + 2a^3 - 6a^2 + 26a - 15) \div (a^2 + 4a - 3)$
13. $(27x^3 + 64) \div (3x + 4)$
14. $(125a^3 + 8) \div (25a^2 - 10a + 4)$
15. $(4x^4 + 11x^2 + 25) \div (2x^2 - 3x + 5)$
16. $(9x^4 + 2x^2 + 1) \div (3x^2 + 2x + 1)$
17. $(a^4 - 4a^3 + 2a^2 + 4a + 1) \div (a^2 - 1 - 2a)$
18. $(x^2 - y^2 - a^2 - 2ay) \div (x - y - a)$
19. $(8a^3 - b^3 + c^3 + 6abc) \div (2a + c - b)$
20. $(m^2 + 6m + 9 - 25x^2) \div (m + 3 - 5x)$
21. $(6x^3 + 11x^2 - 1) \div (3x - 1 + 2x^2)$
22. $(2x^3 + 5x^2 - 7x - 5) \div (2x - 1)$
23. $(6a^2x^2 - 4x^3 - 4ax^3 + x^4) \div (a^2 + x^2 - 2ax)$
24. $(11a^3 - 5a + 12 - 82a^2 + 30a^4) \div (2a - 4 + 3a^2)$
25. $(x^5 + 32y^5) \div (x + 2y)$
26. Divide 1 by $1 - 3x$, carrying the quotient to five terms.
27. What must be the value of m in order that $x^2 - 3 + x$ may be an exact divisor of $2x^2 - 3x^3 + 21x + x^4 + 3m$?
28. Write a formula to indicate the relation between the dividend, D , the divisor, d , the quotient, q , and the remainder, r .

Linear Equations

Solve the following equations:

1. $2x + 3 = 9$
2. $3x - 2 = 13$
3. $7x + 5 = 2x - 15$
4. $4x + 3 = 2x + 8$
5. $7x - 8 = 3x - 2$
6. $4x - 7 = 2x - 2$
7. $4x - 6 = 5x + 1$
8. $2x + 5 = x + 7$
9. $4 - 15x = 11 - 16x$
10. $13x - 6 = 11x + 8$
11. $6x + 17 = 23 + 5x$
12. $6x - 17 = 18 - x$
13. $7x - 5 = 8x - 12$
14. $2x + 17 = 5x + 20$
15. $5x + 72 = 2x + 87$
16. $2x - \frac{7}{2} = 3\frac{1}{2}x + 4$
17. $3(2x - \frac{1}{3}) = 2(x + 5\frac{1}{2})$
18. $27x + \frac{1}{3} = 2(1 + 13x)$
19. $\frac{3}{2}x + 1 = \frac{1}{2}x + 6$
20. $5x - (2x + 3) = 12$
21. $13 + (4x + 7) = 12$
22. $9x - 2(1 + 4x) = 3$
23. $3(x - 6) - 2(4 - x) = 0$
24. $4(4 - 3y) = 2(7 - 4y)$
25. $3(x - 1) - 5(2x + 3) = -32$
26. $2(2x + 5) + 10 = 3(8x - 2)$
27. $6.2n + 3 = 4.7n + 18$
28. $7n - 2.5 = 3n + 5.1$
29. $.6n + 2.3 = 1.75n$
30. $1.3n = 4.5(2n - 3) - 9.6$
31. $(2n - 1)^2 - 4(n - 3)^2 = 25$
32. $(n + 1)(n + 2) - 11 = n^2$
33. $(n + 1)(n + 5) = (n + 2)(n + 3)$
34. $(x + 7)(x + 3) - (x + 6)(x - 1) = 0$
35. $(x - 2)^2 - (x + 3)(x - 5) = 3x + 4$
36. $(2x - 1)^2 + 18 = 4(x - 2)(x - 1) - 1$
37. $x(2x - 3) - 2x(5 + x) = 39$
38. $(x - 4)(x + 8) = 7 - (3 - x)(x + 5)$
39. $(10x - 3)(x + 1) + 5x = (2x + 5)(5x - 4) - 8$
40. $5(x - 1)^2 - 3(x - 2)^2 = (2x - 1)(3 + x) - 6$
41. $(x + 7)(3 - x) = 2(x - 3)(x - 5) - 3(x^2 + 3)$
42. $x^2 = (2x - 3)(x - 7) - (x - 1)(x + 4)$
43. $(x + 5)(2x - 3) - (x - 7)(x + 3) = x(x - 1) + 18$

Factoring

Find the prime factors of each of the following :

1. $a^2 + 2ax$
2. $x^2 - 4xy$
3. $mn + n^2$
4. $3x + x^2$
5. $x^2 - p^2$
6. $4x^2 - 9y^2$
7. $8x^2 - 2y^2$
8. $125m^3 - 80mn^2$
9. $64x^2y - 36y^3$
10. $3mr^2 - 363m^3$
11. $16a^4 - 81x^8$
12. $a^4 - 16b^4$
13. $a^3b^2 - 81a^3$
14. $9a^5 - 36a^3$
15. $64a^{4n} - 25$
16. $49r^{2a} - 25s^{6a}$
17. $3y^{2a} - 6y^a$
18. $x^2 + 2x - 24$
19. $a^2 - 2a - 48$
20. $m^2 - 13m - 48$
21. $x^2 - 7xy + 10y^2$
22. $2x^2 - 10x - 12$
23. $x^3 - 5x^2y - 14xy^2$
24. $r^2 - 2r - 15$
25. $x^2 - 8x - 20$
26. $7x^5 - 7x$
27. $m^2 + 6m - 7$
28. $c^2 - c - 6$
29. $4a^2 - 15ab - 4b^2$
30. $12c^2 + 7c - 12$
31. $6b^2 + 7b - 24$
32. $15a^2 - 7ab - 2b^2$
33. $3y^2 - 7y - 6$
34. $2x^2 + 7x + 3$
35. $2x^2 + xy - 6y^2$
36. $2x^2 - x - 6$
37. $2x^2 + 9x - 35$
38. $4x^2 - 12xy^2 + 9y^4$
39. $9x^2 - 24xy + 16y^2$
40. $4m^2 - 28m + 49$
41. $16x^2 - 120xy + 225y^2$
42. $36m^2 + 60mn + 25n^2$
43. $10a^2 - 27ab + 18b^2$
44. $22x^2 + 71x - 21$
45. $16a^4 - 145a^2 + 9$
46. $ax^3 - 6ax^2y + 9axy^2$
47. $16x^2 - 24x^3y + 56xy^7$
48. $9a^2 + 4bc - 4c^2 - b^2$
49. $a^2 - x^2 - 9y^2 + 6xy$
50. $x^2 - b^2 - a^2 + 2ab$
51. $m^2 - 8mn + 16n^2 - 4r^2$
52. $25x^2 - 20xy + 4y^2 - 9z^2$
53. $x^2 + xy + 5y + 5x$
54. $2a^2 + 2bc - ab - 4ac$
55. $10x^2 - 15x - 2xy + 3y$
56. $a^2 + ab - ac - bc$
57. $12x^2 - 84xy + 147y^2$
58. $15x^3 + 35x^2 - 30x$
59. $9x^{4a} + 30x^{2a} + 25$
60. $x^{2a} - 2x^ay^a + y^{2a}$
61. $x^{2a} - 18x^a + 81$
62. $9x^{4a} - 25y^{2n}$

Reduction of Fractions

Reduce to lower terms :

1. $\frac{x^2 - 4}{x^2 + 4x + 4}$

2. $\frac{x^2 - 9}{x^2 + 6x + 9}$

3. $\frac{x^2 + 3x}{x^2 + 4x + 3}$

4. $\frac{2x + x^2}{6 + x - x^2}$

5. $\frac{16a^3 + 8a^2 - 8a}{16a^4 - 16}$

6. $\frac{3c^2 - 6c}{6c^2 - 24}$

7. $\frac{x^2 + 5x - 24}{x^2 + 6x - 16}$

8. $\frac{x^2 + 8xy}{x^2 + 5xy - 24y^2}$

9. $\frac{4x^2 - 20xy + 25y^2}{10x^2 - 29xy + 10y^2}$

10. $\frac{x^{2a+1} + 2x^{a+1} + x}{x^{2a} - 1}$

11. $\frac{x^{2n} - 1}{x^{2n} - 2x^n - 3}$

12. $\frac{a^{2n+1} - a^{n+1} - 6a}{a^{2n+1} - 6a^{n+1} + 9a}$

Reduce to equivalent fractions having their L. C. D. :

13. $\frac{3}{x-1}, \frac{2}{x+1}, \text{ and } \frac{1}{x^2+1}$

14. $\frac{5}{x^2-4}, \frac{3}{x(x+2)}, \text{ and } \frac{2}{x-2}$

15. $\frac{a}{x^2-ax-2a^2}, \frac{2a}{x^2-4ax+4a^2}, \text{ and } \frac{-3a}{x^2+2ax+a^2}$

16. $\frac{3}{x^2-5x+6}, \frac{2}{x^2-x-2}, \text{ and } \frac{5}{x^2-2x-3}$

17. $\frac{5}{x-2}, \frac{3}{x+3}, \frac{7}{x^2+3x}, \text{ and } \frac{2}{2-x}$

Express as a single fraction :

18. $5 - \frac{2x}{x-3}$

19. $x + \frac{1}{x+1}$

20. $\frac{2}{x-1} - 3$

21. $\frac{5}{a-5} - 2a$

22. $\frac{3a+2b}{a+2b} - 1$

23. $\frac{b-a}{2a+b} + 1$

Addition and Subtraction of Fractions

Simplify the following:

1. $\frac{5}{x-2} + \frac{3}{x-1}$

2. $\frac{2}{3a-7} - \frac{2}{3a+2}$

3. $\frac{5}{a-6} - \frac{3}{a+6}$

4. $\frac{3}{a-b} + \frac{2}{a+b}$

5. $\frac{4}{2a+3} - \frac{2}{a-2}$

6. $\frac{a}{x+y} - \frac{b}{y-x}$

7. $\frac{3}{2a+b} + \frac{2}{b-2a}$

8. $\frac{4x}{x^2-4} + \frac{3x-2}{4-x^2}$

9. $\frac{7}{8a+12b} - \frac{3}{9b-6a}$

10. $\frac{2x}{3x+y} + \frac{4x^2}{y^2-9x^2}$

11. $\frac{a+6}{a-b} + \frac{a-6}{a+b} - \frac{2a^2}{a^2-b^2}$

12. $\frac{a+b}{a-b} - \frac{a-b}{a+b} - \frac{4ab}{a^2-b^2}$

13. $\frac{4}{a-3} + \frac{5}{3+a} - \frac{2}{a^2-9}$

14. $\frac{1}{a+3} + \frac{7}{a^2-a-12} + \frac{a+3}{a-4}$

15. $\frac{a+2b}{a^2-ab} + \frac{2a+b}{a^2-3ab+2b^2} - \frac{a-3b}{a^2-2ab}$

16. $\frac{3x}{x^2+x-2} - \left(\frac{2}{x+2} + \frac{1}{x-1} \right)$

17. $\frac{x+2}{x^2+5x+6} + \frac{x+3}{x^2+7x+12}$

18. $\frac{a+b}{(b-c)(c-a)} + \frac{b+c}{(c-a)(a-b)} + \frac{c+a}{(a-b)(b-c)}$

19. $\frac{x}{xy+y^2} - \frac{y}{xy-x^2} + \frac{x}{x^2-y^2}$

20. $\frac{1}{x-3} + \frac{3}{x+1} + \frac{3}{1-x} - \frac{1}{x+3}$

21. $\frac{1}{x+1} - \frac{1}{1-x} - \frac{2}{x-2} + \frac{2}{x+2}$

Multiplication and Division of Fractions

Perform the indicated operations :

$$1. \frac{a+b}{2} \times \frac{6}{a^2-b^2}$$

$$5. \frac{(x+2)^2}{x^2+8x+15} \times \frac{x+5}{x+2}$$

$$2. \frac{x^2-y^2}{15} \times \frac{6}{x+y}$$

$$6. \frac{3x^2}{7c} \div \frac{2x}{21c^3}$$

$$3. \frac{a+1}{x^2-y^2} \times \frac{x-y}{a^2-1}$$

$$7. \frac{7st^2}{10hk} \div \frac{14t^3}{5h^2k^3}$$

$$4. \frac{2a+b}{x+5y} \times \frac{x^2-25y^2}{4a^2-b^2}$$

$$8. \frac{56}{r+3} \div \frac{8}{9-r^2}$$

$$9. \frac{c^4-d^4}{(c-d)^2} \div \frac{c^2+d^2}{c+d}$$

$$10. \frac{x^2-8x+15}{x-7} \div \frac{x-5}{x^2-10x+21}$$

$$11. \frac{x^2-x-2}{x^2-7x} \div \frac{x^2+2x}{x^2-13x+42}$$

$$12. \frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} \div \frac{(x-y)^2}{x^2-xy}$$

$$13. \frac{a(2a-b)^2}{2a+b} \times \frac{b}{2a^2-ab} \div \frac{b(4a^2-b^2)}{(2a+b)^2}$$

$$14. \frac{2a^2-13a+15}{4a^2-9} \times \frac{2a-1}{a-5} \div \frac{2a-1}{2a+1}$$

$$15. \frac{a^2+4ab+3b^2}{a^2-b^2} \times \frac{a+b}{a-b} \div \frac{a+3b}{(a-b)^2}$$

$$16. \frac{4a^2-9}{9a^2-4} \times \frac{2a-3a^2}{9-9a+2a^2} \div \frac{a^2}{3a-a^2}$$

$$17. \frac{4a^2-4a+1}{9a^2-3a-2} \times \frac{3a^2+7a+2}{2a^2+3a-2} \div \frac{2a^3-a^2}{8a^2-8a+4}$$

$$18. \left(\frac{4}{x+1}\right)^2 \times \frac{(x+1)^3}{x-7} \div \frac{16}{x^2-49}$$

$$19. \frac{6a^2}{a^2-9} \times \frac{a-3}{3a-8} \times \frac{3a^2-a-30}{3a^2+9a} \div \frac{6ax-20x}{3a^2+a-24}$$

Combined Operations with Fractions

Perform the indicated operations and simplify:

$$1. \left(1 + \frac{b}{a-b}\right)\left(\frac{a-b}{a}\right)$$

$$6. \left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{xy}{x-y}\right)$$

$$2. \left(1 - \frac{1}{a}\right) \div \left(1 + \frac{1}{a}\right)$$

$$7. \left(1 + \frac{1}{a-1}\right) \div \left(\frac{1}{a-1}\right)$$

$$3. \left(a - \frac{b}{3}\right) \div \left(\frac{a}{3} - b\right)$$

$$8. \left(x - \frac{1}{x}\right)\left(x - \frac{x}{x+1}\right)$$

$$4. \left(1 + \frac{1}{x}\right)\left(\frac{1}{1+x}\right)$$

$$9. \frac{ab}{a^2 + b^2} \left(1 - \frac{a^4}{b^4}\right)$$

$$5. \left(m - \frac{1}{m}\right) \div \left(\frac{1}{m} + \frac{1}{m^2}\right)$$

$$10. \left(1 + \frac{y^2 - xy}{x^2 + y^2}\right)\left(\frac{x^3 + xy^2}{x^2 - y^2}\right)$$

$$11. \left(4 - \frac{6}{a+1}\right) \div \left(8 - \frac{4a-8}{a^2-1}\right)$$

$$12. \left(\frac{a-1}{a} + \frac{b-1}{b} + \frac{c-1}{c}\right) \div \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

$$13. \left(\frac{2}{x} - \frac{1}{a+x} + \frac{1}{a-x}\right) \div \left(\frac{a+x}{a-x} - \frac{a-x}{a+x}\right)$$

$$14. \left(\frac{2a+3b}{3a-2b} - \frac{3a-2b}{2a+3b}\right)\left(\frac{6a^2+5ab-6b^2}{5ab-a^2}\right)$$

$$15. \left(\frac{6a}{a^2-4} + \frac{3}{2-a}\right) \div \left(\frac{3}{a^2-a-6}\right)$$

$$16. \left(\frac{1}{1-x} - \frac{1}{1+x}\right) \div \left(\frac{1}{1-x^2} - \frac{1}{1+x^2}\right)$$

$$17. \left(\frac{x+2y}{x+y} + \frac{x}{y}\right) \div \left(\frac{x+2y}{y} - \frac{x}{x+y}\right)$$

$$18. \frac{x}{2} \left(\frac{1}{x-y} - \frac{1}{x+y}\right) \left(\frac{x^2-y^2}{x^2y+xy^2}\right)$$

$$19. \left(3a+4 - \frac{4}{a}\right)\left(3a-4 - \frac{4}{a}\right)\left(\frac{a}{9a^2-4}\right)$$

$$20. \left(\frac{x+4}{x^2-x-12} - \frac{x+3}{x^2+x-12}\right) \div \left(1 + \frac{2(x^2-12)}{x^2+7x+12}\right)$$

Fractional Linear Equations

Solve the following:

1. $\frac{5n}{2} = 22 + \frac{7n}{5}$

6. $\frac{7}{3x-1} = \frac{2}{x-1}$

2. $\frac{x}{2} - \frac{x-4}{6} = \frac{5}{3}$

7. $\frac{3}{5n-1} = \frac{1}{2n-1}$

3. $\frac{2x}{3} - \frac{3x-5}{2} = 0$

8. $\frac{4}{7n+3} = \frac{3}{6n+2}$

4. $\frac{1}{x+2} = \frac{1}{3}$

9. $\frac{4x-3}{2x-1} = \frac{4x-7}{2x-5}$

5. $\frac{1}{x+6} = \frac{1}{5}$

10. $\frac{6n-2}{3n+4} = \frac{2n+1}{n+3}$

11. $\frac{x}{3} - \frac{x-3}{3} = 12 - \frac{x+4}{2} - x$

12. $\frac{1}{x-2} + \frac{2}{x-3} = \frac{3}{x-4}$

13. $\frac{7}{x-7} + \frac{3}{2-x} = \frac{4}{x-4}$

14. $3 - \frac{12}{x+1} = 7 - \frac{4x+30}{x+3}$

15. $\frac{5}{x-7} - \frac{7}{x+3} = \frac{5x}{x^2-4x-21}$

16. $\frac{8}{x-2} - \frac{1}{x-5} = \frac{7}{x+1}$

17. $\frac{16x+3}{10} + \frac{5-2x}{5x-1} = \frac{8x-1}{5}$

18. $\frac{4y-1}{2y+3} = \frac{12y^2-13}{4y^2-9} - \frac{2y}{2y-3}$

19. $\frac{3(x-2)}{x+3} = \frac{34-4x}{x^2-9} - \frac{2+3x}{3-x}$

20. $\frac{x}{x+4} - \frac{3x^2+16}{x^2-16} = \frac{2x}{4-x}$

Literal Linear Equations and Formulas

Solve the following equations for x :

1. $ax = m$
2. $ax = bx + c$
3. $7x = ax - c$
4. $\frac{x}{a} = b - c$
5. $\frac{m}{a} = \frac{1}{x} + \frac{2}{a}$
6. $\frac{a}{x} = 1 + \frac{a}{2}$
7. $\frac{3x}{b} + \frac{2x}{a} = 3$
8. $a^2(x - 1) = ax - 1$
9. $2bx + 6a^2 = 4ab + 3ax$
10. $m(x - m) + a(x - a) = 0$
11. $\frac{6}{x - 3a} + \frac{5}{5a - x} = 0$
12. $\frac{x - a}{b} = \frac{x - b}{a}$
13. $\frac{m - x}{n - x} = \frac{m + x}{n + x}$
14. $\frac{x - a}{a + x} = \frac{x + 3b}{2b + x}$

15. Solve $\frac{a - bx}{c} + b = \frac{bc - x}{c}$ for x .

16. Solve $l = a + (n - 1)d$ for n . Find d .

17. Solve $S = \frac{a(b + c)}{2}$ for b . Find a .

18. Solve $S = \frac{ar^n - a}{r - 1}$ for a .

21. Solve $V = \pi r^2 h$ for h .

19. Solve $C = 2\pi r$ for r .

22. Solve $V = \frac{1}{3}bh$ for h .

20. Solve $C = \frac{E}{R + r}$ for r .

23. Solve $C = \frac{en}{R + nr}$ for n .

24. Solve $L = \frac{M - g}{t}$, for t ; for M ; for g .

25. Solve $S = \frac{1}{2}g(2t - 1)$ for t .

26. Solve $T = 2\pi r(h + r)$ for h .

27. Solve $\frac{E}{e} = \frac{R + r}{r}$ for r .

28. Given $F = \frac{2}{3}C + 32$, express C as a function of F .

29. From the formula $R = \frac{gs}{g + s}$, express s as a function of R and g .

Graphs of Formulas and Equations

Graphs of linear functions.

1. Draw a graph to represent the formula for the area of a rectangle, $A = bh$, when $h = 3$ in. From the graph determine: (a) A if $b = 5.5$ in.; (b) b if $A = 19.5$ sq. in.

2. Draw a graph to represent the formula for the perimeter of a square, $p = 4s$. From the graph find p if $s = 8$.

3. Draw a graph of $i = prt$ when $p = 100$ and $r = .04$.

Draw graphs of the formulas:

4. $C = \pi d$ ($\pi = 3.1$) 5. $V = 32t$ 6. $F = \frac{9}{5}C + 32$

7. Write the equation that states that the y -values of a point (x, y) are 3 less than twice the corresponding x -values. Draw a graph of the locus of all such points.

Draw graphs of the equations:

8. $y = 4x + 6$ 9. $3x - 4y = 0$ 10. $4x = 3y + 6$

11. At what point does the graph of $2x = 3y - 6$ cross (a) the x -axis? (b) the y -axis?

Solve graphically the sets of equations:

12. $\begin{cases} 3x + 2y = 0 \\ 5x + 13 = y \end{cases}$ 13. $\begin{cases} 5x - 4y = 6 \\ 2x + y = 3 \end{cases}$ 14. $\begin{cases} 3x + 5y = 1 \\ 2x + 7y = 8 \end{cases}$

Graphs of quadratic functions.

15. Draw a graph of $s = n^2$ from $n = -7$ to $n = +7$. From the graph determine: (a) the square of -3 ; (b) the square root of $12\frac{1}{4}$.

16. The formula $ab = 36$ indicates that the product of two numbers is 36. Represent this by a graph, using: (a) $b = 1$ to $b = 36$; (b) $a = -1$ to $a = -36$.

Draw graphs of the equations:

17. $y = x^2 + 6x - 16$ 19. $x^2 + 3x = y + 2$
 18. $y = x^2 - 6x - 7$ 20. $x^2 - 5x = y$

21. Determine the roots of equations 17 to 20 if $y = 0$.

Systems of Linear Equations

Solve the following equations:

$$1. \begin{cases} 4x - y = 9 \\ 2x - 3y = -23 \end{cases}$$

$$2. \begin{cases} 2x + 7y = 25 \\ 2x + y = 13 \end{cases}$$

$$3. \begin{cases} x - 4y = 1 \\ 3x + 4y = 19 \end{cases}$$

$$4. \begin{cases} 3x - 5y = 1 \\ 2x + 3y = 7 \end{cases}$$

$$5. \begin{cases} 7x - 5y = 31 \\ 4x + 3y = 6 \end{cases}$$

$$6. \begin{cases} 6x - y = 0.9 \\ x + 2y = 0.8 \end{cases}$$

$$7. \begin{cases} 5x + 6y = 51 \\ 6x - 4y = -6 \end{cases}$$

$$8. \begin{cases} 3x + 4y = 36 \\ x + 4 = 4y \end{cases}$$

$$9. \begin{cases} 3x + 4y = 48 \\ 2x + 3y = 26 \end{cases}$$

$$10. \begin{cases} 11x + 2y = 35 \\ 9x - y = 26 \end{cases}$$

$$11. \begin{cases} 3x - 5y = 39 \\ 7x - 22 = 4y \end{cases}$$

$$12. \begin{cases} 7x + 5y = 30 \\ 3x - 4y = 19 \end{cases}$$

$$13. \begin{cases} 2x - 3y = 5 \\ 4x - 3y \end{cases}$$

$$13. \begin{cases} \frac{x+2}{x+6} = \frac{y}{y+3} \end{cases}$$

$$14. \begin{cases} ax + (a+b)y = ab \\ ax + (a-b)y = -ab \end{cases}$$

$$15. \begin{cases} ax + by = m \\ bx + ay = n \end{cases}$$

$$16. \begin{cases} ax + by = b \\ cx - dy = d \end{cases}$$

$$17. \begin{cases} ax + by = 2 \\ abx + aby = a + b \end{cases}$$

$$18. \begin{cases} \frac{3x}{a} + \frac{2y}{b} = 3 \\ \frac{2x}{a} - \frac{6y}{b} = \frac{5}{3} \end{cases}$$

$$19. \begin{cases} ax + by = ab \\ x + \left(1 + \frac{b}{a}\right)y = a \end{cases}$$

$$20. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6} \\ \frac{2}{x} - \frac{1}{y} = \frac{2}{3} \end{cases}$$

$$21. \begin{cases} \frac{5}{x} + \frac{3}{y} = \frac{3}{2} \\ \frac{3}{x} - \frac{2}{y} = \frac{4}{15} \end{cases}$$

$$22. \begin{cases} \frac{1}{x} - \frac{1}{y} = 1 \\ \frac{2}{y} + \frac{1}{x} = 7 \end{cases}$$

$$23. \begin{cases} \frac{9.8}{7x} + \frac{10.5}{5y} = 2 \\ \frac{29.4}{7x} - \frac{6.3}{y} = 1 \end{cases}$$

Squares and Square Roots

Find the indicated squares:

- | | | |
|------------------|---------------------------------------|---------------------------------|
| 1. $(a + x)^2$ | 9. $(5x^2 - \frac{1}{2})^2$ | 16. $(a + \sqrt{b})^2$ |
| 2. $(2a - x)^2$ | 10. $(3x - \frac{y}{2})^2$ | 17. $(3 - \sqrt{y})^2$ |
| 3. $(x + 3a)^2$ | 11. $(\frac{1}{2}x + \frac{1}{3}y)^2$ | 18. $(2\sqrt{x} + 3\sqrt{y})^2$ |
| 4. $(5x - 1)^2$ | 12. $(x - \frac{5}{3}y)^2$ | 19. $(\sqrt{2} + \sqrt{x})^2$ |
| 5. $(7ax + 3)^2$ | 13. $(2x^2 + \frac{1}{2}y)^2$ | 20. $(\sqrt{3} - \sqrt{y})^2$ |
| 6. $(a - 2bc)^2$ | 14. $(x^2 - \frac{3}{4}y^3)^2$ | 21. $(5 + 2\sqrt{x})^2$ |
| 7. $(3m + 7p)^2$ | 15. $(3x - 5y^2)^2$ | 22. $(3\sqrt{2x} - 1)^2$ |
| 8. $(2ax - 3)^2$ | | |

23. By inspection, square each of the following:

- (a) 3.5 (b) 8.5 (c) 12.5 (d) 145 (e)
- $11\frac{1}{2}$

Find the square root of each of the following:

24. $9x^4 - 36x^3 + 38x^2 - 4x + \frac{1}{9}$
 25. $x^6 + 3x^2 - x^4 - 2x^5 + 2x + 1$
 26. $4x^4 + 17x^2 - 12x - 12x^3 + 4$
 27. $a^4 - 6a^3 + 10a^2 - 3a + \frac{1}{4}$
 28. $4c^4 - 4c^3 + 5c^2 - 2c + 1$
 29. $100x^4 + 65x^2 - 100x^3 + 4 - 20x$
 30. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$
 31. $\frac{9}{4}x^4 - x^3 + 15\frac{1}{8}x^2 - \frac{10}{3}x + 25$
 32. $\frac{x^4}{9} + \frac{4}{3}x^3y + 2x^2y^2 - 12xy^3 + 9y^4$
 33. $\frac{a^2}{9} + \frac{2}{15}ab - \frac{2}{3}a + \frac{b^2}{25} - \frac{2}{5}b + 1$

Find to the nearest tenth the square root of:

34. 53 35. 59.29 36. 33 37. 7.5

38. What does the formula $c^2 = a^2 + b^2$ represent? Find c , to the nearest tenth, if $a = 8$ and $b = 5$.

39. A rectangular box is 12 in. long, 5 in. wide, and 10 in. deep. What is the distance, to the nearest tenth of an inch, from one corner to the diagonally opposite corner?

Radicals and Exponents

Simplify each of the following :

1. $3\sqrt{27}$
2. $\sqrt{72}$
3. $\sqrt{98}$
4. $\sqrt{108a}$
5. $\sqrt{196b}$
6. $\sqrt{\frac{18}{5}}$
7. $60\sqrt{\frac{4}{45}}$
8. $32\sqrt{\frac{3}{8}}$
9. $24\sqrt{\frac{2}{3}}$
10. $15\sqrt{\frac{8}{75}}$
11. $\sqrt{a^2 + a^4}$
12. $\sqrt{b^2 + b^2x}$
13. $\sqrt{4a^4 - 20a^2}$
14. $\sqrt{9a^2b^2 - 27}$
15. $\sqrt{25a^2 - 75a^2b^2}$
16. $\sqrt{8\frac{1}{3}}$
17. $\sqrt{1\frac{1}{8}}$
18. $\sqrt{\frac{a^2b}{27}}$
19. $\frac{\sqrt{3}}{\sqrt{5}}$
20. $2\sqrt{108} - 6x\sqrt{5\frac{1}{3}} + \sqrt{3(8x - 1)^2}$
21. $30\sqrt{\frac{1}{2}} - \frac{9}{2}\sqrt{8} + 9\sqrt{\frac{169}{2}}$
22. $\sqrt{27x^2} - \sqrt{3(7x - 2)^2} + 6\sqrt{\frac{4x^2}{3}}$
23. $6a\sqrt{\frac{2}{3}} - \frac{b}{3}\sqrt{54} + \sqrt{24b^2}$
24. $\sqrt{16a^3 - 48a^2b} - \sqrt{9ab^2 - 27b^3} + \sqrt{a^3b^2 - 3a^2b^3}$
25. $\sqrt{2}(3 - 5\sqrt{2})$
26. $\sqrt{3}(2\sqrt{3} - \sqrt{12})$
27. $(6 - 2\sqrt{5})^2$
28. $(5\sqrt{8} + 3\sqrt{6})^2$
29. $(3\sqrt{2} - 2\sqrt{3})^2$
30. $(5\sqrt{5} - 4)(5\sqrt{5} + 8)$
31. $(3\sqrt{5} - \sqrt{2})(2\sqrt{5} + 4\sqrt{2})$
32. $(\sqrt{27} - \sqrt{12})(3\sqrt{8} + \sqrt{18})$

Find the value, to the nearest tenth, of :

33. $\frac{2\sqrt{3} - 5}{\sqrt{3}}$
34. $\frac{3\sqrt{2} + 2\sqrt{3}}{\sqrt{6}}$
35. $\frac{\sqrt{27} - \sqrt{48}}{\sqrt{12}}$

Find the value of :

36. $8^{\frac{2}{3}} - 9(27)^0 + 9^{\frac{3}{2}} - 16^{\frac{1}{4}} + (5)^{\frac{1}{2}}(5)^{\frac{3}{2}} + 169^{\frac{1}{2}} - 64^{\frac{2}{3}} + 81^0$
37. $x^2 + x^{\frac{3}{2}} - x + x^{\frac{1}{2}} - 2x^0$ when $x = 4$
38. $5x^{\frac{4}{3}} + 3x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + (2x)^0$ when $x = 8$
39. $x^2 + 2x - 4$ when $x = 3\sqrt{5} + 1$
40. $x^2 - 2x - 11$ when $x = 1 - 2\sqrt{3}$
41. Express ex. 36 to 38 without fractional exponents.

Quadratic Equations

Solve the following equations and check the results :

- | | |
|-------------------------|--|
| 1. $x^2 - 10x - 39 = 0$ | 12. $x^2 - 6x = 10$ |
| 2. $2x^2 + x - 15 = 0$ | 13. $x^2 - \frac{1}{6}x - \frac{1}{3} = 0$ |
| 3. $x^2 - 20 = 8x$ | 14. $3x^2 - 7x - \frac{1}{4} = 0$ |
| 4. $3x^2 - 2x = 8$ | 15. $x + 1 = \frac{1}{2}x^2 - \frac{1}{8}x$ |
| 5. $2x^2 + x - 21 = 0$ | 16. $5x^2 - 6x = 1\frac{2}{5}$ |
| 6. $3x^2 - 11x = 70$ | 17. $\frac{1}{3}x + \frac{3}{x} + 5 = 7\frac{1}{2}$ |
| 7. $2x^2 - 3x = 9$ | 18. $x^2 - 21 = 2x$ |
| 8. $8x^2 - 2x - 3 = 0$ | 19. $\frac{1}{2}(y + 1) - \frac{y}{3}(2y - 1) = \frac{2}{3}$ |
| 9. $6x^2 + x = 1$ | 20. $2x^2 + 5 = 10(x + 1)$ |
| 10. $6x^2 - 2 = x$ | |
| 11. $2x^2 - 4x = 5$ | |

Find the value of x to the nearest tenth :

- | | |
|-------------------------|-------------------------|
| 21. $x^2 = 3x + 5$ | 26. $x^2 - 10x + 2 = 0$ |
| 22. $3x^2 - 16x = 32$ | 27. $2x^2 - 5x - 4 = 0$ |
| 23. $2x^2 + 4 = 7x$ | 28. $3x^2 + x - 3 = 0$ |
| 24. $2x^2 + 6x - 3 = 0$ | 29. $5x^2 + 2x = 4$ |
| 25. $3x^2 = 2 + 4x$ | 30. $7x^2 - 5x = 1$ |

Solve the equations :

- | | |
|--|--|
| 31. $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3$ | 36. $\frac{x+2}{x-1} + \frac{x-4}{2x} = \frac{7}{3}$ |
| 32. $\frac{x}{4} - \frac{x}{x+4} = \frac{x-1}{6}$ | 37. $(x+a+b)^2 = (a-b)^2$ |
| 33. $\frac{x^2}{x-1} = \frac{1}{x-1} + 10$ | 38. $x^2 + bx + bc + cx = 0$ |
| 34. $\frac{2}{x-1} - 1 = \frac{3x-1}{x+3}$ | 39. $x^2 - 2ax + 4ab = 2bx$ |
| 35. $\frac{3x}{x-2} + \frac{2}{2-x} = \frac{2}{x+3}$ | 40. $x^2 - 2cx = 4a - c^2$ |
| | 41. $x^2 + 5ax = 14a^2$ |
| | 42. $16a^2x^2 + 1 = 10ax$ |
| | 43. $21x^2 - 3a^2 = 2ax$ |
| | 44. $3x^2 - 5ax = 112a^2$ |

Systems of Quadratic Equations

Solve the following equations :

$$1. \begin{cases} 2x^2 - y^2 = 14 \\ x - y = 11 \end{cases}$$

$$2. \begin{cases} 2x - y = 5 \\ x^2 - 4y^2 = 5 \end{cases}$$

$$3. \begin{cases} x - y = 2 \\ x^2 - 2y^2 = 7 \end{cases}$$

$$4. \begin{cases} 2x - y = 12 \\ x^2 + 2xy = 5 \end{cases}$$

$$5. \begin{cases} 2x - 5y = 0 \\ x^2 - 3y^2 = 13 \end{cases}$$

$$6. \begin{cases} 3x + 2y = 5 \\ xy + 3 = 6x \end{cases}$$

$$7. \begin{cases} 2x^2 + y^2 = 9 \\ 3x - y = 7 \end{cases}$$

$$8. \begin{cases} xy + x^2 = 24 \\ y - 3x = -4 \end{cases}$$

$$9. \begin{cases} x^2 - 3y^2 = 13 \\ 1 + 2y - x = 0 \end{cases}$$

$$10. \begin{cases} x(y + 1) = 4 \\ 3x = 2y \end{cases}$$

Many of the questions and problems that follow have been taken from recent examinations.

True and False Statements

State which of the following statements are *true* and which are *false*. If a statement is false, correct it.

1. A coefficient of a term is a factor of the term.
2. The difference between two negative numbers may be a positive number.
3. If the numerator and the denominator of a fraction are equal, the value of the fraction is 0.
4. The fourth root of a number is the square of the square root of the number.
5. If both terms of a fraction are positive, decreasing the denominator of the fraction decreases the value of the fraction.
6. If the same number is added to both terms of a fraction, the value of the fraction is not changed.
7. Every quadratic equation has two roots.
8. The square root of any positive number is less than the number.
9. The product of two binomials may be a binomial.

10. One of the roots of the equation $3x^2 + 8x - 3 = 0$ is $\frac{1}{3}$.
11. The equation $\frac{5x}{16} - \frac{4x}{5} = 0$ has 2 for a root.
12. The roots of the system $\begin{cases} x^2 - y = 4 \\ x + y = 16 \end{cases}$ are $\begin{cases} x = 4, -5 \\ y = 21, 12 \end{cases}$
13. In a radical expression the index of the root to be found is always an even number.
14. The dividend is equal to the product of the divisor by the quotient and the result diminished by the remainder.
15. The graph of a linear equation is a straight line.
16. If both members of an equation are divided by the same number, the quotients are always equal.
17. If both members of an equation are multiplied by the same number, the products are always equal.
18. If four quantities are in proportion, the product of the first and the third is equal to the product of the second and the fourth.
19. The prime factors of $36x^2 - 9y^2$ are $6x - 3y$ and $6x + 3y$.
20. $4 \times 8 - 6 + 2 + 8 + 4 - 31 = 0$.
21. $\sqrt{16a^2 + 9b^2} = 4a + 3b$.
22. $\sqrt{25x^2 - 16} = 5x - 4$.
23. If a is greater than b , then $(a - b)^2 = (b - a)^2$.
24. (a) $\frac{\sqrt{18}}{\sqrt{2}}$ is a rational number.
(b) $5.333\frac{1}{3}$ is a rational number.

Linear Equations — Geometric Problems

1. Find each angle of a triangle if the first angle is twice as large as the second angle and the third angle is equal to the sum of the other two angles.
2. Find each angle of an isosceles triangle if each of the base angles is twice the size of the vertex angle.

3. The perimeter of an isosceles triangle is 16 in. Find each side if one of the equal sides is 2 in. longer than the base.

4. Two angles are complementary. Find each angle if their difference is 15° .

5. Find two supplementary angles if one angle is 30° less than the other angle.

6. Find an angle such that its supplement is 50° larger than twice its complement.

7. Two angles of a certain triangle are in the ratio 6 to 5 and the third angle is the difference between the other two angles. How large is each angle?

8. The perimeter of a rectangular field is 160 rd. Find the dimensions if they are to each other as 3 to 5.

9. The sum of the base and the altitude of a triangle is 32 in. and their ratio is as 5 to 3. Find the area of the triangle.

10. Find the area of a trapezoid if the sum of its altitude and two bases is 24 in. and the ratio of the altitude and the bases is as 3 to 4 to 5.

Miscellaneous Problems

1. Divide 60 into two parts such that 3 times one part shall equal twice the other part.

2. Separate 72 into two parts such that twice one part exceeds 8 times the other part by 4.

3. A classroom has 27 seats, some single and the rest double. If the room seats 42 students, how many double seats are there?

4. Two boys together had 150 marbles. One arranged his in groups of 5 and the other, in groups of 8. They had 24 groups between them. How many marbles did each have?

5. Frank is 10 yr. older than his brother Henry; 10 yr. ago he was 3 times as old. How old is each now?

6. At a certain place the longest day exceeds the shortest night by 7 hr. 40 min. How long is each?

7. A man bought a number of sheep at \$6.50 each. Four of them died and he sold the remainder at \$7.75 each. How many sheep did he buy if he received \$19 more than he paid for the sheep?

8. The cost of a horse and wagon was \$225. The cost of the horse lacked \$25 of being 4 times the cost of the wagon. What was the cost of each?

9. A farmer sowed 13 acres with wheat and oats, obtaining a total yield of 496 bu. If the wheat yielded 32 bu. per acre and the oats, 40 bu. per acre, how many acres of each did he plant?

10. A man sold 2 acres more than $\frac{3}{5}$ of his farm and had 4 acres less than $\frac{1}{2}$ of it left. How many acres did he sell?

11. The amounts possessed by Albert and John are in the ratio of 7 to 5. If Albert gives John \$15, they will have equal amounts. How much has each?

12. An estate of \$5700 is to be divided among a mother, a son, and a daughter. The mother is to get \$2500 more than the son, and the daughter is to get twice as much as the son. Find the amount each will receive.

13. Divide \$1285 among John, Alice, and Henry so that John has \$25 more than $\frac{5}{8}$ as much as Alice has, and Henry's money is to Alice's as 4 to 15.

14. The denominator of a fraction is double its numerator. If the numerator is increased by 3 and the denominator is decreased by 4, the value of the fraction becomes 1. Find the original fraction.

15. What number must be added to the numerator and the denominator of the fraction $\frac{9}{17}$ so that the resulting fraction shall equal $\frac{3}{4}$?

16. Mr. Rice wishes to divide \$1600 among his three sons so that the second son will receive $\frac{3}{5}$ as much as the first son and the third son will receive an amount equal to the other two. Find the share of each.

17. Chester was offered for his bicycle a sum equal to $\frac{5}{9}$ of what it cost him. He sold it later for $\frac{3}{4}$ of what it cost him, thereby receiving \$7 more than he was offered at first. How much did it cost?

18. A man walks at the rate of 4 mi. an hour. How far can he walk into the country and ride back in an automobile at the rate of 20 mi. an hour if he must be back home in 3 hr. from the time he starts?

19. Two automobiles are 276 mi. apart and start at the same time to travel toward each other. They travel at rates differing by 5 mi. an hour and meet in 6 hr. Find the rate of each.

20. Two automobiles leave Albany at the same time. The first one travels east and the other goes west. In 6 hr. they are 297 mi. apart. Find the rate of each if their rates are as 4 to 5.

21. Mr. James leaves Omaha at 8 A.M. in an automobile going at a rate of 27 mi. an hour. Three hours later his son starts on a motorcycle to overtake him, traveling at a rate of 39 mi. an hour. How far must the son travel before he overtakes his father?

22. At the rate of 3 mi. an hour uphill and 4 mi. an hour downhill, a man walked $17\frac{1}{2}$ mi. in 5 hr. How much of the distance was uphill?

23. Fred and James ride their bicycle around a 350-yard circular track. Fred goes at the rate of 450 yd. a minute and James at the rate of 280 yd. a minute. Both start at the same time and go in the same direction. How long will it take Fred to catch up with James and pass him?

24. A father was 3 times as old as his son 12 yr. ago, and 3 yr. hence the father will be twice as old as his son. How old is each now?

25. Arthur is 5 yr. older than his sister and $\frac{1}{6}$ as old as his father. The sum of the ages of all three is 51 yr. How old is Arthur?

26. The ratio between Mrs. Clark's age and her daughter's age is 5 to 1; but 3 yr. ago the ratio was 9 to 1. How old is Mrs. Clark now?

27. A purse contains 18 coins, some quarters and the rest dimes. If the coins are worth \$2.40, how many are there of each kind?

28. Mary opened her bank and found \$2.50 in quarters, nickels, and dimes. There were $\frac{1}{2}$ as many quarters as nickels and $\frac{1}{3}$ as many dimes as nickels. How many coins of each kind were there?

29. A farmer employed a boy on the condition that the boy was to receive \$2 for each day that he worked and to pay \$1 board for each day that he was idle. At the end of 20 da. the boy received \$34. How many days had he worked?

30. Mrs. Corwin has \$9000 invested, part at $5\frac{1}{2}\%$ and the rest at 6%. If she receives a yearly income of \$525, how much is invested at each rate?

31. Mr. Adkins invested money in bonds that paid 5%, $5\frac{1}{2}\%$, and 6%. The amount invested at 5% was \$2000 less than that invested at 6%, and the amount invested at $5\frac{1}{2}\%$ was 3 times as great as that at 5%. If his annual income was \$670, how much money did Mr. Adkins have invested at each rate?

32. A man sold two houses for \$23,000. On one he gained 25% of the cost and on the other he lost 20%. If his net gain was \$1000, what was the cost of each house?

33. Thirty men agreed to be responsible for a dinner. Later 5 men withdrew from the agreement and the rest had to pay \$.50 more each. What was the price agreed on?

Problems with Two Unknown Numbers

1. A man finds that he can get 2 tons of coal and 3 loads of wood for \$45 or he can get 3 tons of coal and 5 loads of wood for \$70. How much should he pay for 7 loads of wood and 10 tons of coal?

2. In a barnyard there are pigs and hens having in all 50 heads and 140 feet. How many of each kind are there?

3. The tens' digit of a two-digit number exceeds the units' digit by 3. The sum of the digits is $\frac{1}{7}$ of the number. Find the number.

4. A number is expressed by two digits whose sum is 8. If 6 times the tens' digit is added to the number, the digits are interchanged. Find the number.

5. If 3 is added to the numerator and 5 is subtracted from the denominator of a certain fraction, the value of the fraction is $\frac{3}{7}$. If 2 is subtracted from the numerator and 3 is added to the denominator, the value becomes $\frac{1}{5}$. Find the fraction.

6. How many pounds of 45-cent coffee and how many pounds of 35-cent coffee must be mixed so that 50 lb. of the mixture may sell at \$.42 a pound?

7. In the equation $y = ax + b$, it is known that a and b are constants and that $y = 4$ when $x = 2$ and $y = 13$ when $x = 5$. Find the value of a and of b .

8. One man and 2 boys can do in 12 days a piece of work that could be done in 6 days by 3 men and 1 boy. How long would it take a boy to do the work? a man?

9. Frank and Gus can do a piece of work in 30 days. At the end of 18 days, however, Gus quits and Frank finishes the work alone in 20 days more. Find the time in which each could do the work alone.

10. If 5 yd. of cloth and 2 yd. of silk cost \$18.50 and 3 yd. of cloth and 5 yd. of silk cost \$22.50, what is the price per yard of each material?

Problems Involving Quadratic Equations

1. Find a number such that if $\frac{2}{3}$ of the number is subtracted from the square of the number, the result is 75.

2. Find the number whose square diminished by 20 is equal to 8 times the number.

3. Find the side of a square such that if twice its perimeter is added to the number of square units in its area, the result is 20.

4. The perimeter of a certain rectangle is 248 ft. and its area is 3700 sq. ft. Find the length of each side.

5. A rectangle is $1\frac{1}{2}$ times as long as it is wide. If 2 ft. are taken from each dimension, the new rectangle thus formed has $\frac{2}{3}$ the area of the original one. Find the area of each rectangle.

6. A clothier purchased a number of overcoats for \$720, paying the same amount for each. He sold them for \$40 each, clearing an amount equal to the cost of 8 coats. How many coats did he buy?

7. If the speed of a railway train should be lessened 4 mi. an hour, the train would be $\frac{1}{2}$ hr. longer in going 180 mi. Find the rate of the train.

8. An automobile goes to a place 84 mi. distant and returns in 7 hr. The average rate of speed going is 7 mi. an hour less than that returning. Find the rate going and the rate returning.

9. Two men, A and B, can dig a trench in 20 da. It would take A alone 9 da. longer to dig it than it would take B alone. How long would it take B alone?

10. Separate a line 20 in. long into two parts such that the ratio between the whole line and the longer part shall equal the ratio of the longer part to the shorter part.

11. At Christmas Mr. Jones distributed \$180 equally among a number of his friends. Mr. Hunt distributed the same amount among 30 people less, but he gave each person \$3 more than Mr. Jones did. How much did Mr. Jones give each person? Mr. Hunt?

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